Calculus 3

Vector Analysis

Steve Trettel

2025-04-22

Table of contents

Preface

$$\iint_{\Sigma} \nabla \times \vec{F} \cdot d\vec{S} = \int_{\partial\Sigma} \vec{F} \cdot d\vec{s}$$

This is the set of course notes for my Calculus 3 course (math 211) at the University of San Francisco.

Part I.

Space

In this part we acquaint ourselves with the mathematics of n-dimensional spaces. Such spaces are described using n-tuples of real numbers

$$(x_1, x_2, \dots, x_n)$$

and are indespensible when discussing positions in real, *physical* space (especially in dimension 3 and 4 for space and spacetime). But they are also indespensible to mathematicians working on other problems, where high dimensional spaces are used to track the behavior of complex objects.

We learn to differentiate between *points* (which measure position) and *vectors* (which give a direction and magnitude), and cover the operations of dot and cross product which are essential to the geometry of vectors.

Finally, we use our knowledge of points and vectors to construct formulas for shapes in 2,3 and higher dimensional spaces; from lines planes and spheres to more complicated objects.

1. Dimensions

(Relevant Section of the Textbook: 12.1 Three Dimensional Coordinate Systems, and 10.3 Polar Coordinates)

To do calculus in higher dimensions, we need first a precise mathematical language that will allow us to *describe* these spaces. And that language begins with a foundational, but straightforward definition: the n-tuple.

Definition 1.1 (n-Tuple)**.** An n tuple of real numbers is an ordered list of real numbers. For example, a 2-tuple like $(3, 7)$ is often called an *ordered pair*. Tuples are sometimes written horizontally and sometimes vertically, depending on convenience. Various styles of brackets are used on tuples, depending on the author and usage. Below are examples of a 3-tuple, a 4-tuple, and a 7-tuple in several styles:

$$\langle 0, -12, 0.3\rangle \qquad \begin{pmatrix} 3 \\ -7 \\ \pi \\ 4 \end{pmatrix} \qquad [1, 2, 3, 4, 5, 6, 7]$$

Just as numbers represent a location on the line, tuples can be used to represent locations in space. When we use them as such, we call the entire n-tuple a *point*, and we call each of the entries a *coordinate*.

Definition 1.2 (Point)**.** A *point* is an n-tuple when it is being used to represent a location in space.

You are already familiar with this from single variable calculus, where we use orderd pairs (x, y) to represent points of the 2-dimensional plane $\mathbb{R}^2$. By extension, we can use three-tuples (x, y, z) to represent points in the physical space around us. But what about even bigger tuples, like the 7-tuple $[1, 2, 3, 4, 5, 6, 7]$? What kind of space does this represent a point in? This is a point in a *seven* dimensional space!

Definition 1.3. The dimension of a space is the number of coordinates needed to describe a point in it.

A plane is 2-dimensional, but so is the surface of a sphere: if your friend called you and gave you two numbers - their latitude and longitude - you could precisely locate them on the Earth's surface.

$$(37.7749^\circ N, 122.4194^\circ W)$$

The space around us is three dimensional because if I wanted to direct you to my apartment I would need to give you not only the two street intersections (two numbers, specifying a point on the earth's surface) but also the *floor I live on* (the height above the surface).

$$(1^{st}\text{Street}, 3^{rd}\text{Ave}, 4^{th}\text{Floor})$$

But the space-*time* we live in is *four-dimensional* because if we wanted to meet for lunch I would need to give you four numbers, my position in space and also when to meet, so that we do not miss each other, you thinking luch is at 11 and I thinking noon.

$$(1^{st}\text{Street}, 3^{rd}\text{Ave}, 4^{th}\text{Floor}, 12\text{pm})$$

Thus, there are direct phyiscal reasons to consider calculus in two, three and four dimensions. And, our best physical theories of the world at human scales (classical mechanics) are written in this language. Understanding weather, planetary motion, fluid flow, and black holes requires a solid grounding in multivariable calculus. But, since the real world is only four dimensional, does that mean there is no need for the calculus of 7, or 13, or 132,234,453 dimensional space?

1.1. Higher Dimensions and Configurations

All the spaces we have talked about so far represent *physical space*, but mathematics is allows us to be much more general than this. Imagine you are designing a tin can, and you want to start by thinking of the *space of possibilities*: what are all the possible shapes of a cylindrical can? As such a can is fully determined by its radius and its height, we can think of these as being two *coordinates*, and expressing a particular can by an ordered pair (r, h). Thus, the *space of possible cans* is 2-dimensional!

What about the space of L-spaced desks? What is the dimension of this space? This space has 5-dimensions: the length and width of each of the two sides of the desk, and also its height.

$$(\ell_1, \ell_2, w_1, w_2, h)$$

But where really mind-blowing numbers of dimensions begin to arise is in the study of *data*. Imagine you are modeling the conditions in the San Francisco bay, and you take a measurement of the sea height for every square kilometer. The bay has an area of 4000 square kilometers, so this means your datapoint has $4,000$ numbers in it! Your approximation to the bay's surface is a point in four thousand dimensional space!

Or, consider an image taken by a digital camera: for simplicity assume the image is in black and white, and 10 megapixels. This means each pixel is determined by a single number (how light or dark the pixel is), and the image has 10 million pixels, so it's encoded by ten million numbers! That means even simple images are worked with mathematically as *points* in a space with millions of dimensions.

1.2. Cartesian Coordinates

What do the actual numbers in the n-tuple mean? In the examples above we have been implicitly using xyz or length-width-height coordinates: the first number tells you the distance left/right, the second back/forth and the third up/down. These are called *Cartesian Coordinates* after the mathematician-philosopher Rene Descartes. While this sort of thinking comes from *physical* space, it is useful to help give a concrete picture even to *non-physical spaces* such as the space of soup cans, or the space of images.

Definition 1.4 (Cartesian Coordinates)**.** Cartesian coordinates starts by choosing n perpendicular lines in n dimensional space: for example the x and y axes in the plane, or the x, y, z axes in $\mathbb{R}^3$. A point in space is given coordinates (a, b, c) if it lies at distance a along the first axis, b along the second, and c along the third.

Here's an animation showing a point in 3D space, and its components along the x, y and z axes.

https://stevejtrettel.site/code/2023/vector-components

Perhaps the most famous theorem of geometry is the Pythagorean Theorem, which tells us how to compute distance in the cartesian coordinates on the plane:

Theorem 1.1. *The distance of the point* (a, b) *from the point* $(0, 0)$ *in the plane is*

$$\mathrm{dist} = \sqrt{a^2 + b^2}$$

Once we know this theorem is true in $\mathbb{R}^2$ (thanks, Pythagoras!) we can use this proof to extend it to higher dimensions!

Theorem 1.2. *In $\mathbb{R}^3$, the distance of (x, y, z) from the origin is*

$$\text{dist} = \sqrt{x^2 + y^2 + z^2}$$

In general, if $(x_1, \dots, x_n)$ is a point in $\mathbb{R}^n$, its distance from the origin is

$$\text{dist} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Given this description of distance, we can give a precise description of circles and spheres: a circle is the set of points which are all a fixed distance (the radius) from a fixed point (the center). The *unit circle* is the set of points distance 1 from the origin. Similarly, the unit sphere is the set of points distance 1 from the origin in 3-dimensional space.

In more generality, we can define a *hypersphere* in any higher dimension the same way: by taking the set of points which are a fixed distance from the origin in that space!

Definition 1.5. The unit circle (sometimes called the unit 1-sphere) is the set of points in $\mathbb{R}^2$ given by

$$x^2 + y^2 = 1$$

The unit sphere (sometimes called the unit 2-sphere) is the set of points in $\mathbb{R}^3$ satisfying

$$x^2 + y^2 + z^2 = 1$$

The unit 3-sphere is the set of points in *four dimensional space* satisfying

$$x^2 + y^2 + z^2 + w^2 = 1$$

And so on...

Exercise 1.1. Why is the circle called a 1-sphere, and the sphere in $3D space called the 2-sphere?

We can also use cartesian coordiantes to describe other simple shapes, like lines. In $\mathbb{R}^2$ with cartesian coordinates, the x-axis is given by the set of points $\{(x, 0)\}$: that is, every point on the x-axis has y-coordinate equal to zero.

Definition 1.6 (Coordinate Axes and Planes)**.** The x axis in the plane is given by the equation $y = 0$, and the y axis by the equation $x = 0$. Similarly in $\mathbb{R}^3$, the xy plane is given by the equation $z = 0$, the yz plane by the equation $x = 0$, and the xz plane by the equation $y = 0$.

A point can be *projected* onto a coordinate axis or plane by setting that coordinate to zero. The resulting point is the *closest* point on that line or plane to the original point. This makes it a relatively straightforward calculation to find the distance from a point to a coordinate axis/plane!

Here's some practice problems:

1.3. Other Coordinate Systems

While the majority of our class will occur in Cartesian coordinates as they are the first coordinate system everyone must master, we will at times consider a couple of other interpretations of n-tuples, which make describing certain systems with circular or spherical symmetry easier. The first of these may already be familiar from earlier calculus classes: polar coordinates on the plane.

Definition 1.7 (Polar Coordinates)**.** A point (r, θ) in polar coordinates on the plane, for $r > 0$ and $\theta \in [0, 2\pi)$ represents the point which lies at a distance r from the origin, and makes an angle of θ with the positive x-axis.

From this description we can convert a point in polar coordinates to cartesian coordinates (x, y) using trigonometry:

Definition 1.8. The conversion from polar coordinates (r, θ) to cartesial coordiantes (x, y) is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$$

Using polar coordinates simplifies many things when circles are involved: for example, the equation of the unit circle $x^2 + y^2 = 1$ becomes the much simpler equation $r = 1$ in polar coordinates!

Using polar coordinates for x, y in the 3-dimensional space (x, y, z) gives a coordinate system called *cylindrical coordinates.*

Definition 1.9 (Cylindrical Coordinates)**.** A 3-tuple (r, θ, z) represents a point in $\mathbb{R}^3$ using cylindrical coordiantes where the position in the xy plane is given by the polar coordinates (r, θ) and the height above the xy plane is given by z. The conversion to carteisan coordinates is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix}$$

Finally we will come across a third coordinate system in this class: *spherical coordinates*. This represents 3-dimensional space starting with a collection of concentric spheres. Don't worry too much about this now, we will come back to it in some weeks! I've only placed it here for your future reference.

Definition 1.10 (Spherical Coordinates)**.** A 3-tuple (ρ, θ, ϕ) represents a point in $\mathbb{R}^3$ using spherical coordiantes where ρ is the distance from the origin, θ is the angle with the positive x axis (as in polar coordiantes) and ϕ is the angle with the positive z-axis. The conversion to cartesian coordinates is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho\cos\theta\sin\phi \\ \rho\sin\theta\sin\phi \\ \rho\cos\phi \end{pmatrix}$$

1.4. Videos

1.4.1. Cartesian Coordinates

https://youtu.be/iBgOoaeLUcM

1.4.2. Higher Dimensions

https://youtu.be/yjUy9kevd3Y

https://youtu.be/qo0KKSXWW4E

< https://youtu.be/-hGgDALLaWU >

< https://youtu.be/1I3gXuvLuqI >

1.4.3. Distances

https://youtu.be/GJDi4_Af_lI

2. Vectors

(Relevant Section of the Textbook: 12.2 Vectors)

We have talked about one fundamental use of n-tuples of real numbers: describing *posititions* in space. But they also play a foundational role in the theory of *vectors*, which help us measure not *locations* but *directions*.

Definition 2.1 (Vector)**.** A vector is a *directed line segment*, an object that stores both a *length* (called its *magnitude*), and a *direction*. You may draw a vector as a directed line segment, or a little arrow in space.

Example 2.1 (Points vs Vectors)**.** Which of the following quantities are positions? Which are vectors?

- Where I parked my car.
- The wind hitting me in the face
- The location of mars in the solar system.
- The velocity of mars in the solar system.
- Gravity's acceleration

Exercise 2.1. Come up with some of your own scenarios that are measured using: 2-d points. 3-d vectors. 4d points, 4d vectors.

To work with vectors, we need a means of writing them down using numbers. One idea is to use the same *Cartesian coordinate system* discussed in the previous chapter, to express a vector in *components*.

Definition 2.2 (Vector Notation)**.** To help avoid confusing *points* and *vectors*, we will try to use different notations for the two. For a point, we use a simple letter, like p whereas for a vector we either make it bold like $\mathbf{u}$ or decorate it with an arrow, $\vec{u}$.

When using cartesian coordinates, we write a point inline with round brackets, $p = (1, 2, 3)$ whereas for a vector we use angle brackets, $\vec{u} = \langle 1, 2, 3\rangle$. We also sometimes write a vector as a *column* of numbers, instead of a row to save space, or for other stylistic reasons: in this case we just use round brackets for ease of typesetting.

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

Definition 2.3 (The Zero Vector)**.** The zero vector in an n dimensional space is the vector with no magnitude and no length. In coordinates, this is the n-tuple of all zeroes: $\langle 0,0,0\rangle$

2.1. Arithmetic of Vectors

Definition 2.4 (Vector Addition)**.** Vector addition is defined to give the *combined effect* of two vectors: if $\vec{u}$ and $\vec{v}$ are vectors, $\vec{u}+\vec{v}$ is defined geometrically by the diagonal of the parallelogram with sides $\vec{u}$ and $\vec{v}$. This is equivalent to the vector formed by stacking $\vec{u}$ and $\vec{v}$ on one another head-to-tail, in either order.

In cartesian coordinates, this is

$$\begin{pmatrix}x\\y\\z\end{pmatrix}+\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}x+a\\y+b\\z+c\end{pmatrix}$$

PICTURE

Vector addition can be used to describe complicated motion in terms of simpler pieces. Indeed, this idea was used by the ancient greeks in their planetary model, where the complicated motion of objects in the heavens was modeled as a combination of various circular motions (or epicycles). Below is an animation showing their model of Mars' motion about the earth, decomposed in terms of a sum of three circles.

https://stevejtrettel.site/code/2020/epicycle-mars/

Definition 2.5 (Scalar Multiplication)**.** If $\vec{u}$ is a vector and c is a number, the vector $c\vec{u}$ is defined to be the vector pointing in the same direction as $\vec{u}$, but c times as long. In Cartesian coordinates,

$$c\langle u_1,u_2,u_3\rangle=\langle cu_1,cu_2,cu_3\rangle$$

Because we think of real numbers as being the kind of things that can *scale* vectors, we often call them **scalars**.

Definition 2.6 (Linear Combination)**.** A linear combination of a list of vectors is a new vector made by taking a sum of scalar multiples of the original list.

For example, if $\vec{u}=\langle 1,2\rangle$ and $\vec{v}=\langle 3,4\rangle$, then the following is a linear combination:

$$\vec{w}=7\begin{pmatrix}1\\2\end{pmatrix}-2\begin{pmatrix}3\\4\end{pmatrix}=\begin{pmatrix}7-6\\14-8\end{pmatrix}=\left(1,6\right)$$

In this calculation you saw a bit of vector arithmetic: it works just like the arithmetic of numbers, one coordinate at a time.

Theorem 2.1 (Vector Arithmetic)**.** *Let $\vec{u}, \vec{v}$ and $\vec{w}$ be vectors, and c, k be scalars. Then:*

$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \qquad \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$

$$\vec{u} + \vec{0} = \vec{u} \qquad \vec{u} + (-\vec{u}) = \vec{0}$$

$$c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v} \qquad (c + k)\vec{u} = c\vec{u} + k\vec{u}$$

$$(ck)\vec{u} = c(k\vec{u}) \qquad 1\vec{u} = u$$

2.2. Coordinate Bases

Cartesian coordinates are built from a collection of perpendicular axes. Each of these axes has a *direction* that we call a *standard basis direction*

Definition 2.7 (Standard Basis)**.** For $\mathbb{R}^2$ the standard basis vectors are the vectors $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$, pointing along the positive direction of the x and y axes.

For $\mathbb{R}^3$, the standard basis vectors are teh vectors $\langle 1, 0, 0,$ $\langle 0, 1, 0\rangle$ and $\langle 0, 0, 1\rangle$ pointing along the positive direction of the x, y and z axes respectively.

In general n-dimensional space, the n basis vectors are the vecctors which have all 0s as coordinates except a single 1. The vector whose 1 is in the i^{th} coordinate is called the i^{th} *basis vector.*

For example, $\langle 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0\rangle$ is the 4^{th} standard basis vector of 12-dimensional space. In two and three dimensions we give each of the bases a unique letter to aid in readability, instead of dealing with messy subscritps for only a handful of symbols.

Definition 2.8 (Standard Basis in $\mathbb{R}^2$ and $\mathbb{R}^3$)**.** In $\mathbb{R}^2$ we write the standard basis vectors as

$$\hat{\imath} = \langle 1, 0\rangle \qquad \hat{\jmath} = \langle 0, 1\rangle$$

In $\mathbb{R}^3$ we continue this, writing

$$\hat{\imath} = \langle 1, 0, 0\rangle \qquad \hat{\jmath} = \langle 0, 1, 0\rangle \qquad \hat{k} = \langle 0, 0, 1\rangle$$

We can use these standard basis vectors to express any vector in space. For example, in $\mathbb{R}^3$ every vector is some amount in the $\hat{\imath}$ direction, some amount in the $\hat{\jmath}$ direction, and some amount in the $\hat{k}$ direction. This means we can write any vector as a *linear combination* of these:

$$\vec{u} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

https://stevejtrettel.site/code/2023/vector-components

2.3. Magnitude and Direction Information

One common use for vectors is to give *directions* to get from one point to another: that is, given points p, q in space, we want a vector starting at p and ending at q.

PIC

This vector encodes the magnitude and direction information of "if you are at p and you walk this amount in this direction, you'll arrive at q". We can construct such a vector

Definition 2.9 (Vector from Two Points)**.** The *displacement vector* from a point $p = (p_1, p_2, p_3)$ to a point $q = (q_1, q_2, q_3)$ is

$$\vec{d} = q - p = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{pmatrix}$$

And, analogously in other dimensions.

Thus, the vector $\langle x, y, z\rangle$ is the displacement vector from the *origin$ to the* point* (x, y, z). Its length (or magnitude) is just the distance from the origin to its other endpoint, whih we know from the pythagorean theorem.

Definition 2.10 (Magnitude)**.** The magnitude of the vector $\vec{u} = \langle u_1, u_2, u_3\rangle$ is given by the pythagorean theorem:

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

And, analogously in other dimensions.

A vector of length 1 is called a *unit vector.* We think of these as measuring *purely direction* just as we think of numbers as measuring *purely length.*

Definition 2.11 (Unit Vector in Given Direction)**.** If $\vec{v}$ is a nonzero vector, the unit vector in direction $\vec{v}$ is denoted $\hat{v}$, and is calculated by dividing $\vec{v}$ by its own magnitude:

$$\hat{v} := \frac{1}{\|\vec{v}\|}\vec{v}$$

Exercise 2.2 (Unit Vectors in Given Directions)**.** Find a unit vector in the direction of $\langle 1, 2, 3, 4\rangle$.

Exercise 2.3 (Unit Vectors in Given Directions)**.** Find a vector of length 2 in the direction of $\langle 1, 1, 1, 1, 1, 1, 1, 1\rangle$. *Hint: find a unit vector in this direction. What happens to its length if you scalar multiply it by* 2*?*

2.4. Videos

https://youtu.be/fNk_zzaMoSs

https://youtu.be/etsFVs354GM

https://youtu.be/3V_3dnO-0lo

https://youtu.be/F6iTnOoJ9as

2.4.1. Video Tutorial Series

Here's a short series of videos going through the different vector properties discussed above:

https://youtu.be/51vgIfdBlAk

https://youtu.be/MoHMw0ZO7bU

https://youtu.be/lulSApFPw1M

https://youtu.be/MpN8BIci-Ys

https://youtu.be/DXB1PWq8Dg0

3. Operations

(Relevant Sections of the Textbook: 12.3 The Dot Product, and 12.4 The Cross Product)

3.1. The Dot Product

The dot product is an operation which takes in two vectors and outputs a single number. We will use it as a tool to measure several things, but it's perhaps easiest to learn by just diving right in with the definition.

Definition 3.1 (Dot Product)**.** If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ then the dot product of $\vec{u}$ and $\vec{v}$ is the scalar

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

And similarly in other dimensions, if $\vec{u}, \vec{v}$ are vectors in n dimensions, then

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^{n} u_i v_i$$

For example, the dot product of $\langle 1, 2 \rangle$ and $\langle 3, 4 \rangle$ is $1 \times 3 + 2 \times 4 = 3 + 8 = 11$, and the dot product of $\hat{\imath} - \hat{\jmath}$ and $2\hat{\imath} - \hat{\jmath} + 3\hat{k}$ can be computed by either (1) converting to coordinate notation, or (2) pairing up coefficients and multiplying.

$$(\hat{\imath} - \hat{\jmath}) \cdot (2\hat{\imath} - \hat{\jmath} + 3\hat{k}) = (1 \times 2) + (-1 \times -1) + (0 \times 3) = 3$$

We will unpack a lot more of the geometry hidden inside of this simple defintion soon, but a first thing to notice is that the *magnitude* of a vector can be recovered from its dot product with itself.

Theorem 3.1. *If $\vec{u}$ is any vecotr, then the magnitude of $\vec{u}$ can be calculated via*

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$$

Exercise 3.1. Check that this works for the vector $\langle x, y, z \rangle$ in $\mathbb{R}^3$.

The dot product is built out of the multiplication and addition of ordinary numbers, so it also inherits alot of their algebraic properties:

Theorem 3.2 (Properties of the Dot Product)**.** *If $\vec{u}, \vec{v}$ and $\vec{w}$ are vectors and c, k are scalars then*

$$\vec{0} \cdot \vec{u} = \vec{0} \qquad \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} + \vec{w} + \vec{v} \cdot \vec{w} \qquad c(\vec{u} \cdot \vec{v}) = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$$

3.1.1. Measuring Angles

We've already seen that the dot product of a vector with itself measures the (square of the) magnitude of that vector. But dot products are also able to recover *direction* information as well.

Theorem 3.3 (Angles and the Dot Product)**.** *If $\vec{u}, \vec{v}$ are vectors, their dot product is related to the angle θ between them via*

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$

Re-arranging this, we get a formula that computes the angle between two vectors using only vector operations!

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

Exercise 3.2. What is the angle between $\langle 1, 2\rangle$ and $\rangle 3, 4\rangle$ in radians?

What is the angle between $\hat{\imath} + \hat{\jmath}$ and $\hat{\jmath}$ in degrees?

Definition 3.2 (Orthogonality)**.** Two vectors are called orthogonal if the angle between them is 90 degrees, or $\pi/2$ radians. Since $\cos(\pi/2) = 0$, this means that two vectors are orthogonal if and only their dot product is zero.

Note that the zero vector dotted with any other vector always gives zero, so we say the zero vector is *orthogonal* to every other vector.

3.1.2. Projections

One very useful application of the dot product is to help measure "how much of vector $\vec{v}$ is pointed in direction $\hat{u}$?"

The *scalar projection* of $\vec{v}$ onto the unit vector $\hat{u}$ is the dot product $\vec{v}\cdot\hat{u}$. If $\vec{u}$ is not a unit vector to start with, we first make it into a unit vector by dividing by its magnitude, to find the scalar projection: \$$\vec{v}\cdot\hat{u} = \vec{v}\cdot\frac{\vec{u}}{|\vec{u}|}$

Definition 3.3 (Scalar Projection)**.** The scalar projection of $\vec{v}$ onto $\vec{u}$ is

$$\text{comp}_{\vec{u}}(\vec{v}) = \frac{\vec{v}\cdot\vec{u}}{|\vec{u}|}$$

Exercise 3.3 (Scalar Projection)**.** What is the scalar projection of $\langle 1, 2\rangle$ onto $\langle 3, 4\rangle$?

What is the scalar projection of $\langle 4, 3, -2\rangle$ onto $\langle 0, 1, 0\rangle$?

This tells us *how much* of a vector is pointed in a given direction, and so the answer is a scalar, or number. Oftentimes it is useful to compute a *vector* from this, whose direction is in the direction of $\vec{u}$, and magnitude is the scalar we just computed. This vector can be thought of as the projection of $\vec{v}$ onto $\vec{u}$, or the *shadow* of $\vec{v}$ on the line spanned by $\vec{u}$.

PICTURE

Definition 3.4 (Vector Projection)**.** The *vector projection* of a vector $\vec{v}$ onto a vector $\vec{u}$ is the *scalar projection* times the *unit vector in direction $\vec{u}$. In symbols:

$$\begin{aligned}\text{proj}_{\vec{u}}(\vec{v}) &= \text{comp}_{\vec{u}}(\vec{v})\\ &= \left(\frac{\vec{v}\cdot\vec{u}}{\|\vec{u}\|}\right)\frac{\vec{u}}{\|\vec{u}\|}\\ &= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}\end{aligned}$$

Exercise 3.4 (Vector Projection)**.** Find the vector projection of $\langle 1, 1, 2\rangle$ onto the vector $\langle -2, 3, 1\rangle$.

3.1.3. Standard Basis

Theorem 3.4. *If $\vec{v} = \langle a, b, c\rangle$ is a vector, its scalar projections onto the three standard basis vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ are*

$$\mathrm{comp}_{\hat{\imath}}(\vec{v}) = \vec{v} \cdot \hat{\imath} = a$$

$$\mathrm{comp}_{\hat{\jmath}}(\vec{v}) = \vec{v} \cdot \hat{\jmath} = b$$

$$\mathrm{comp}_{\hat{k}}(\vec{v}) = \vec{v} \cdot \hat{k} = c$$

This confirms our notions of "amount" and "angle" make sense with our original interpretation of the vector $\langle a, b, c\rangle$ as being an amount a in the direction $\hat{\imath}$, b in the direction $\hat{\jmath}$, and c in the direction $\hat{k}$.

Theorem 3.5 (Direction Angle)**.** *The angles α, β, γ a vector $\vec{v} = \langle a, b, c\rangle$ makes with the direction of the standard basis $\hat{\imath}, \hat{\jmath}, \hat{k}$ respectively are*

$$\cos\alpha = \frac{\vec{v} \cdot \hat{\imath}}{\|\vec{v}\|} = \frac{a}{\|\vec{v}\|}$$

$$\cos\beta = \frac{\vec{v} \cdot \hat{\jmath}}{\|\vec{v}\|} = \frac{b}{\|\vec{v}\|}$$

$$\cos\gamma = \frac{\vec{v} \cdot \hat{k}}{\|\vec{v}\|} = \frac{c}{\|\vec{v}\|}$$

3.2. Cross Product

In two dimensions, given one vector $\vec{v} = \langle a, b\rangle$ it is easy to find an orthogonal vector to it: $\vec{v}^{\perp} = \langle b, -a\rangle$ for example.

In 3D, the analogous problem is more difficult: given two vectors, how do we find a third vector that is orthgonal to them *both at once*? A nice solution to this is given by the **cross product**.

Definition 3.5 (Cross Product)**.** The cross product of $\vec{u} = \langle u_x, u_y, u_z\rangle$ and $\vec{v} = \langle v_x, v_y, v_z\rangle$ is

$$\vec{u} \times \vec{v} = \langle u_y v_z - u_z v_y, u_x v_z - u_z v_x, u_x v_y - u_y v_x\rangle$$

https://stevejtrettel.site/code/2023/cross-product

Exercise 3.5 (Orthogonality of the Cross Product)**.** Check that if $\vec{u} = \langle a, b, c\rangle$ and $\vec{v} = \langle x, y, z\rangle$ are two vectors, that $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

While there are many vectors orthogonal to $\vec{u}$ and $\vec{v}$, this particular choice has some very nice mathematical properties. In particular, it gets it's name because it acts algebraically a lot multiplication:

Theorem 3.6 (Properties of the Cross Product)**.** *Let $\vec{u}, \vec{v}, \vec{w}$ be vectors, and k a scalar. Then*

$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

$$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$$

$$(k\vec{u}) \times v = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$$

However, the cross product has two **very important differences** from regular multiplication: the order matters!

Theorem 3.7 (Non-Commutativity of the Cross Product)**.** *If $\vec{u}$ and $\vec{v}$ are vectors, then*

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

Not only does the order that you place the vectors in the product matter, but if you are doing more than one cross product, the order in which you perform them matters as well!

Theorem 3.8 (Non-Associativity of the Cross Product)**.** *If $\vec{u}$, $\vec{v}$ and $\vec{w}$ are vectors, then*

$$(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$$

3.2.1. Computing the Cross Product

Definition 3.6 (2×2 determinants)**.** The determinant of a 2×2 matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Using this notation, we can recast the definition of the cross product from Definition 3.5 to look like

$$\vec{u} \times \vec{v} = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} \hat{\imath} - \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} \hat{\jmath} + \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \hat{k}$$

Note the *minus sign* on the second term (compare with Definition 3.5 to see where a -1 was factored out). This formula can in turn be written even more compactly using the definition of a 3×3 determinant:

Definition 3.7 (3 × 3 determinants)**.** The determinant of a 3 × 3 matrix is

$$\begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \end{vmatrix} = x\begin{vmatrix} b & c \\ e & f \end{vmatrix} - y\begin{vmatrix} a & c \\ d & f \end{vmatrix} + z\begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

Putting it all together,

Definition 3.8. The cross product of $\vec{u} = \langle u_x, u_y, u_z\rangle$ and $\vec{v} = \langle v_x, v_y, v_z\rangle$ is

$$\begin{aligned} \vec{u}\times\vec{v} &= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} \\ &= \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix}\hat{\imath} - \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix}\hat{\jmath} + \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}\hat{k} \end{aligned}$$

While the cross product is specific to three dimensions, later in the course we will sometimes apply it to 2-dimensional vectors, where we think of a vector $\vec{v} = \langle x, y\rangle$ in $\mathbb{R}^2$ as being the vector $\vec{v} = \langle x, y, 0\rangle$ in $\mathbb{R}^3$. In this case, the cross product of two planar vectors has a rather simple formula (for instance, we know it must point directly along the z-axis!)

Definition 3.9 (Planar Cross Product)**.** If $\vec{u} = \langle a, b\rangle$ and $\vec{v} = \langle c, d\rangle$ are two vectors in $\mathbb{R}^2$, their cross product, *when thought of as vectors in the xy plane of* $\mathbb{R}^3$ is

$$\vec{u}\times\vec{v} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}\hat{k} = \langle 0, 0, ad - bc\rangle$$

3.2.2. Geometry of the Cross Product

The main property of the cross product is that it's a third vector that its orthogonal to the two vectors you start with. If you did Exercise 3.5, you've already proven this theorem:

Theorem 3.9 (Orthogonality of the Cross Product)**.** *The vector $\vec{u}\times\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.*

Howver there is an entire line of vectors orthogonal to $\vec{u}$ and $\vec{v}$. Which of these is the cross product? We need to understand its *magnitude* and its *direction.*

Theorem 3.10. *The magnitude of the cross product $\vec{u}\times\vec{v}$ is given in terms of the magnitudes of $\vec{u}$ and $\vec{v}$, and the angle θ between them:*

$$\left|\vec{u}\times\vec{v}\right| = |\vec{u}||\vec{v}|\sin\theta$$

This formula, *base* times *height* times the *sine of the angle between them* may be familiar as the area of a parallelogram. That gives an even more useful interpretation of the cross product's length:

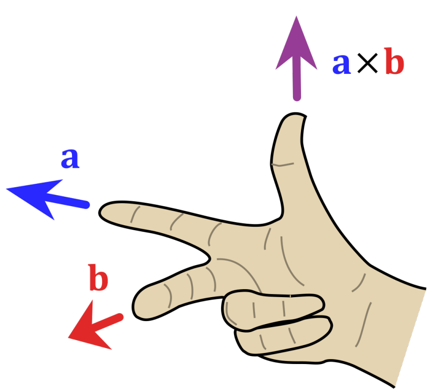
Theorem 3.11. *The magnitude of the cross product $\vec{u} \times \vec{v}$ is the area of the parallelogram spanned by $\vec{u}$ and $\vec{v}$.*

This has an immediate corollary: if $\vec{u}$ and $\vec{v}$ are parallel the parallelogram they span collapses onto a line. And lines have zero area, so the cross product must have zero length! That is, it must be the vector of all zeroes.

Corollary 3.1. *If $\vec{u}$ and $\vec{v}$ are parallel, then their cross product is the* zero vector*.*

This gives us the *magnitude* information, but what about the direction? There are two possible directions a vector could point if it must be perpendicular to the plane containing $\vec{u}$ and $\vec{v}$

Theorem 3.12 (The Right Hand Rule)**.** *The* right hand rule *is a mnemonic to help remember the direction of the cross product. If you take your right hand and align your palm with the vector $\vec{u}$ and then curl your fingers towards the vector $\vec{v}$, your thumb will point in the direction $\vec{u} \times \vec{v}$.*



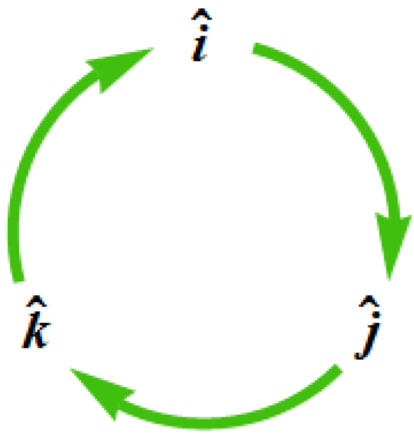
3.2.3. The Standard Basis

It's often useful to know the value of the cross product on the standard basis vectors, to speed up some computations and avoid the lengthy formula in Definition 3.8.

Theorem 3.13.

$$\hat{\imath} \times \hat{\jmath} = \hat{k} \quad \hat{\jmath} \times \hat{k} = \hat{\imath} \quad \hat{k} \times \hat{\imath} = \hat{\jmath}$$

This is helpfully illustrated by a diagram:



Recalling that the cross product changes sign when you reverse the order of the inputs, its possible to compute the rest of the possible stanard basis products:

$$\hat{\imath} \times \hat{\jmath} = \hat{k} \implies \hat{\jmath} \times \hat{\imath} = -\hat{k}$$

In terms of the diagram, this means if you read "backward" along an arrow, you insert a minus sign.

3.3. The Triple Product

If you have three vectors, its possible to combine the dot and cross product to get a single number: take the cross product of two of them to get another vector, then dot that result with the third.

Definition 3.10 (Triple Product)**.** The scalar triple product of the vectors $\vec{u}, \vec{v}$ and $\vec{w}$ is defined by

$$\vec{u} \cdot (\vec{v} \times \vec{w})$$

Writing out the cross product using Definition 3.8, we see that this is actually a big 3×3 determinant:

Theorem 3.14 (Triple Product)**.** *The triple product of vectors $\vec{u}, \vec{v}, \vec{w}$ is equal to the 3×3 determinant*

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Just as the 2×2 determinant measures the *area of a parallelogram*, this 3×3 determinant measures the *volume of a parallelpiped*

Theorem 3.15 (Triple Products & Volume)**.** *The magnitude of the triple product of* $\vec{u}, \vec{v}, \vec{w}$ *is the volume of the parallepiped spanned by these vectors.*

3.4. Videos

3.4.1. Dot Products

From the "Calculus Blue" series by Prof Robert Ghrist.

https://youtu.be/kw-TTdfJwY0

https://youtu.be/nRh8XqWEoj4

https://youtu.be/LxSMhIUaIc4

https://youtu.be/Dsa2MX2grMg

A video reviewing the basic definition of the dot product.

https://youtu.be/Tfuu7iSxGIA

The following two videos review the theory and an example for using the dot product to find the angle between vectors.

https://youtu.be/Tfuu7iSxGIA

https://youtu.be/4WxniMJYySc

A short video on the definition of orthogonality.

https://youtu.be/tGYvaabMbYA

Real world applications of the dot product

https://youtu.be/TBpDMLCC2uY?si=8GL1huhqLnIDOwQy

3.4.2. Cross Products

https://youtu.be/_tXqoAehVR0

https://youtu.be/-734RN3BqPk

https://youtu.be/UAAQlqMCc8c

3.4.3. Triple Products

https://youtu.be/McWNGB1USQE

4. Shapes

(Relevant Sections of the Textbook: 12.5 Lines & Planes, and 12.6 Cylinders and Quadric Surfaces)

Lines and planes are given by *linear* equations: involving only the coordinate variables (x, y, z,etc), constants, and addition.

4.1. Lines

Definition 4.1 (Implicit Lines in $\mathbb{R}^2$)**.** An implicit line in the plane is an equation of the form

$$ax + by = c$$

. When $b \neq 0$ this can be put into $y = mx + b$ form as $y = -\frac{a}{b}x + \frac{c}{b}$.

Theorem 4.1 (Normal Direction to a Line)**.** *The implicit line $ax + by = c$ is orthogonal to the vector $\vec{n} = \langle a, b \rangle$.*

How can we confirm this? Find *two points* along the line, subtract them to get their *direction vector*, then take the dot product of this with $\langle a, b \rangle$ - show its zero!

Finding the direction of a line by finding two points on it is tedious, so luckily once we know the above fact we don't need to do this any more! Its easy to figure out the *drection* of an implicit line: if its orthogonal to $\langle a, b \rangle$, then it points in the direction $\langle -b, a \rangle$ (recall this vector is itself orthogonal to $\langle a, b \rangle$).

Definition 4.2 (Parametric Lines)**.** A parametric line is a function of the form

$$\ell(t) = p + t\vec{v}$$

This passes through the point p and is in the direction $\vec{v}$.

https://stevejtrettel.site/code/2023/parameterized-line

Theorem 4.2 (Line Segment Between Two Points)**.** *If $\vec{p}$ and $\vec{q}$ are two points in space, the line segment between them can be parameterized by the following equation for $t \in [0, 1]$.*

$$\begin{aligned}\vec{\ell}(t) &= \vec{p} + t(\vec{q} - \vec{p}) \\ &= (1 - t)\vec{p} + t\vec{q}\end{aligned}$$

4.2. Planes

Definition 4.3 (Implicit Planes I). A plane through the point $\vec{p} = \langle x_0, y_0, z_0 \rangle$ with normal vector $\vec{n} = \langle a, b, c \rangle$ is determined by the scalar equation

$$\vec{n} \cdot (\langle x, y, z \rangle - p) = 0$$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Distributing and collecting all the constants on the right hand side, we see planes are given by simple, *linear* equations in three variables.

Definition 4.4 (Implicit Planes II). A plane in $\mathbb{R}^3$ is specified by a scalar equation of the form

$$ax + by + cz = d$$

The original derivation also allows us to easily read off the normal vector to a plane:

Theorem 4.3 (Normal Direction to a Plane). *The implicit plane $ax + by + cz = d$ is orthogonal to the vector of coefficients $\vec{n} = \langle a, b, c \rangle$*

To write down a parametric line, we chose a point p in space, and a direction $\vec{v}$. We then added scaled versions of $\vec{v}$ to p, which traced out a line. We can do something analogous with a plane, except we pick two direction vectors, and have *two* scaling parameters

Definition 4.5 (Parametric Planes). A parametric plane through the point p containing the vectors $\vec{u}$ and $\vec{v}$ is given by the function

$$P(s, t) = p + s\vec{u} + t\vec{v}$$

https://stevejtrettel.site/code/2023/parametric-plane

What is the normal vector to a parametric plane? We know two vectors on the plane $\vec{u}$ and $\vec{v}$, so their cross product must be orthogonal to the plane.

Theorem 4.4 (Normal to a Parametric Plane). *If $P(s, t) = p + s\vec{u} + t\vec{v}$ is a parametric plane, the vector*

$$\vec{n} = \vec{u} \times \vec{v}$$

is a normal vector to it.

Exercise 4.1 (Implicit Equation from a Parametric Plane)**.** Consider the following parametric plane:

$$P(s,t) = \begin{pmatrix}1\\2\end{pmatrix} + s\begin{pmatrix}-1\\3\end{pmatrix} + s\begin{pmatrix}1,-1\end{pmatrix}$$

What is a point that it passes through? What is its normal vector? What's an implicit equation for this plane?

4.3. Circles and Spheres

Definition 4.6 (Circle)**.** The circle C of radius R centered at a point p in the plane is the set of all points which lie at distance R from p.

$$C = \{q \in \mathbb{R}^2 \mid \mathrm{dist}(p,q) = R\}$$

We can use the distance function on the plane to come up with an implicit formula for the circle:

Theorem 4.5 (Implicit Circle in $\mathbb{R}^2$)**.** *The circle of radius R centered at $p = (p_x, p_y)$ is given by the implicit equation*

$$(x - p_x)^2 + (y - p_y)^2 = R^2$$

Can we also find a *parametric* description of the circle? Using the trigonometric identity $\cos^2\theta + \sin^2\theta = 1$, we see that if $x = \cos\theta$ and $y = \sin\theta$ then (x, y) must lie on the unit circle about the origin!

Theorem 4.6 (Parametric Unit Circle$)**.** *The unit circle centered at* $(0, 0)$ *can be parameterized as*

$$\vec{C}(t) = \begin{pmatrix}\cos t\\ \sin t\end{pmatrix}$$

for $t \in [0, 2\pi]$.

Translating and scaling this:

Theorem 4.7 (Parametric Circle in $\mathbb{R}^2$)**.** *The circle of radius R centered at $p = (p_x, p_y)$ is given by the parametric equation*

$$R\,C(t) + p = \begin{pmatrix}R\cos t\\ R\sin t\end{pmatrix} + \begin{pmatrix}p_x\\ p_y\end{pmatrix}$$

Definition 4.7 (Sphere)**.** The sphere S of radius R centered at $p \in \mathbb{R}^3$ is the set of points in 3-dimensional space that lie at distance R from p:

$$S = \{q \in \mathbb{R}^3 \mid \mathrm{dist}(p, q) = R\}$$

Theorem 4.8 (Implicit Sphere in $\mathbb{R}^3$)**.** *The sphere of radius R centered at $p = (p_x, p_y, p_z)$ is given by the implicit equation*

$$(x - p_x)^2 + (y - p_y)^2 + (z - p_z)^2 = R^2$$

It is also possible to make a parametric equation of a sphere. Since the surface of a sphere is two dimensional we will need *two parameters* much like we did for planes. The expression looks a bit complicated the first time you see it, and while we will not need it until later in the course, it appears below for completeness.

Theorem 4.9 (Parametric Unit Sphere)**.** *The sphere of radius* 1 *centered at* $(0, 0, 0)$ *can be parameterized by*

$$S(u, v) = \begin{pmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v) \end{pmatrix}$$

For $u \in [0, 2\pi]$ *and* $v \in [0, \pi]$.

Below is a program illustrating the parameric unit sphere: the rectangle on the bottom is the space of *parameters* (the $u - v$ coordinates), and you can track the red, blue lies and the black point between the inputs in the rectangle and the outputs in the sphere.

Try going into the menu and messing with the sliders under *domain*, to see what the two different angles do.

https://stevejtrettel.site/code/2023/tangent-plane

Just as for the circle, we can take this parameterization for the unit sphere and use it to find one for *any sphere* by scaling and translating it:

Theorem 4.10 (Parametric Sphere)**.** *The sphere of radius R centered at $p = (p_x, p_y, p_z)$ is given by*

$$R\,S(u, v) + p = R\begin{pmatrix} \cos(u)\sin(v) \\ \sin(u)\sin(v) \\ \cos(v) \end{pmatrix} + \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

4.4. Other Shapes

The equation $x^2 + y^2 = 1$ is the implicit equation for a circle in the plane. But what shape does this determine in three dimensions?

Example 4.1 (Cylinder)**.** A *cylinder* is a set of points in $\mathbb{R}^3$ which project orthogonally onto a circle in some plane. That is, a "stack of circles" along some axis.

The easiest examples are cylinders around *coordinate axes*: For example, $x^2 + y^2 = 1$ is a **circle** in $\mathbb{R}^2$ but is a **cylinder** in $\mathbb{R}^3$, as the x, y coordinates make a circle but the z coordinate is free to be anything.

Similarly, $y^2 + z^2 = 4$ is a circle of radius 2 around the x-axis.

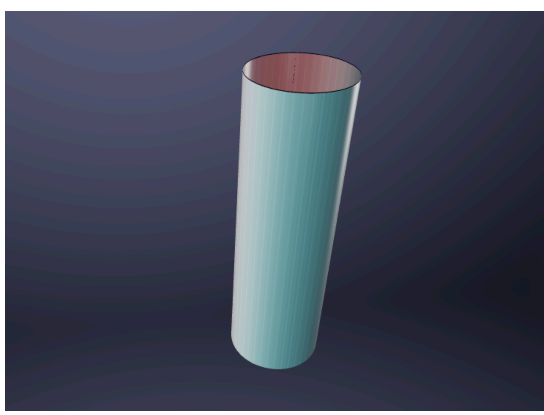


Figure 4.1.: The cylinder $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

In general, the implicit equation of any 2-d shape can be used in 3 dimensions to describe a "stack" of those two dimensional shapes along the direction of the variable that's missing from the formula.

Example 4.2. The equation $y = x^2$ traces out a **Parabolic cylinder**, that is, a stack of parabolas in the z-direction.

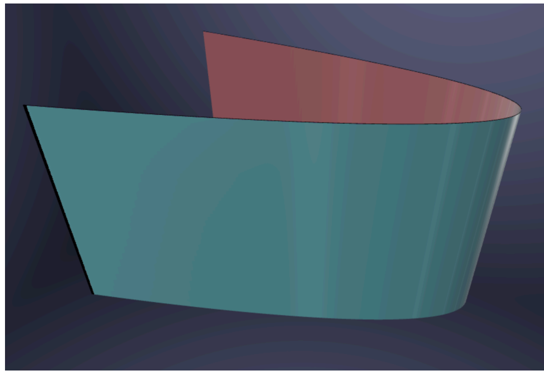


Figure 4.2.: The parabolic cylinder $y = x^2$ in $\mathbb{R}^3$.

Other shapes that will be useful are ellipsoids, which are squashed spheres:

Example 4.3 (Ellipsoid)**.** An ellipsoid is given by the implicit equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

This is similar to the equation of an ellipse but with one more variable.

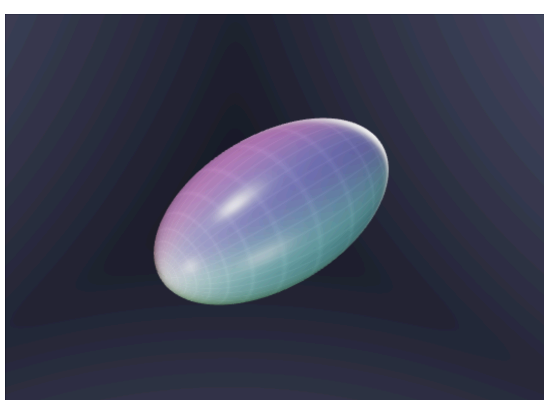


Figure 4.3.: The ellipsoid $\frac{x^2}{2} + y^2 + z^2 = 1$ in $\mathbb{R}^3$.

Example 4.4 (Paraboloid)**.** A paraboloid is the surface $z = x^2 + y^2$, or a stretched version $z = ax^2 + by^2$ where a and b are positive. Its cross sections are circles (in the first case) and ellipses (in the second).

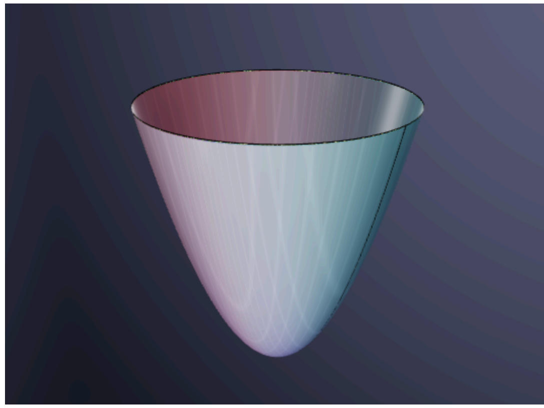


Figure 4.4.: The paraboloid $z = x^2 + y^2$ in $\mathbb{R}^3$.

Example 4.5 (Saddle)**.** A saddle shaped surface, or *hyperbolic paraboloid* is a surface of the form $z = x^2 - y^2$ or a scaling of it. In this form, it has a cross section like a upwards-facing parabola along the x axis, and a downwards-facing parabola along the y axis.

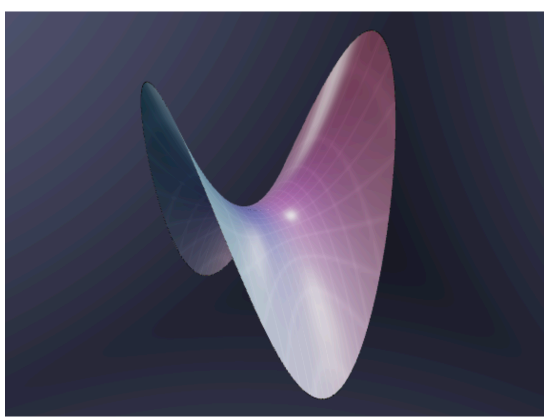


Figure 4.5.: The hyperbolic paraboloid $z = x^2 - y^2$ in $\mathbb{R}^3$.

4.4.1. Helpful Videos

A discussion of implicit vs parametric equations (the Calculus Blue series)

https://youtu.be/UuYXPaac7gU?si=f8AQSdcnkNORl8-1

Here's some videos reviewing some of the techniques we learned in class: first, a couple involving planes.

https://youtu.be/2sZKZHyaQJ8?si=Ld79HXRv12YfQaL2

https://youtu.be/rL9UXzZYYo4?si=6IwekLHYirpagr1Y

And secondly, a review of precalculus material on putting circle euqations into standard form.

https://youtu.be/u_39J-syjB0?si=rJYGjMXPwy8W5eYe

Examples of quadratic surfaces

https://youtu.be/5y1bhGsYG8o?si=aYYIh8kOkN6b-KPn

Part II.

Curves

5. Parameterization

(Relevant Section of the Textbook: 13.1 Vector Functions and Space Curves)

Definition 5.1 (Plane Curve)**.** A plane curve is a function $\vec{c}: \mathbb{R} \to \mathbb{R}^2$. Written in the coordinates (x, y) of $\mathbb{R}^2$, a plane curve can be expressed using two *coordinate functions*

$$\vec{c}(t) = (x(t), y(y))$$

We've seen examples of plane curves already, for instance parametric lines (like $\ell(t) = (2t - 1, 3t + 4)$) and parametric circles, (like $c(t) = (2\cos(t) + 1, 2\sin(t) - 1)$).

Definition 5.2 (Space Curve)**.** A space curve is a function $\vec{r}: \mathbb{R} \to \mathbb{R}^3$. Written in the coordinates (x, y, z) of $\mathbb{R}^3$, a spade curve can be expressed using three *coordinate functions*

$$\vec{c}(t) = (x(t), y(y), z(t))$$

https://stevejtrettel.site/code/2023/parametric-curve-animation

We see curves like this in everyday life - watching a bird fly through the air we see its position changing in time, so its x, y, and z components are all chaninging in time $(x(t), y(t), z(t))$. Likewise, watching your car driving on a GPS, we see the car's position changing - its latitude and longitude are functions of time $\text{car}(t) = (\text{lat}(t), \text{long}(t))$. Indeed this is how we usually will use plane and space curves to trace out *positions* as a function of time. But it is also often useful to think of a curve as being traced out by a little arrow based at the origin (the vector picture, vs the position picture). This is particularly helpful when trying to *build* curves for yourself, as you can think about adding on terms, scalar multiplication, etc.

Here's a graphing calculator for curves, so you can try making some of your own. Can you make this draw a circle in the yz plane?

https://stevejtrettel.site/code/2023/parametric-curve

Parametric curves are used in animation, physics, and engineering. In this visualization below, each little blob is animated using a parametric curve: this could trace out a school of fish swarming at sea, for instance.

https://stevejtrettel.site/code/2022/integral-curves

5.1. Parameterization Tips

Studying the *properties* of parametric curves falls squarely within mathematics, and we will spend much time soon developing the calculus to do so. But *creating* parametric curves is more art than science: it really helps to build up some intution for a few basic examples, and then learn how to combine them and modify them to produce new and more interesting curves. I encourage you to follow along in the discussions below using the graphing calculator at the top of the page.

Example 5.1 (Parametrizing The Graph of a Function)**.** If $y = f(x)$ is a function, its graph consists of the y value $f(x)$ whenever the x-value is x. That means, the graph of f consists of the points $(x, f(x))$ in the plane. Expressed yet a third way, a parametric equation that traces out the graph is given by

$$\vec{c}(t) = (t, f(t))$$

Recall that an *implicit* equation gives a relationship of y and x that are satisfied by an equation. Some of these express functions like $y = x^2$, but others do not, for instance $x^2 + y^2 = 1$. Oftentimes given an implicit equation its desirable to *parameterize it*: to find a way to trace out the curve as a function of t. There's no single way to do this, and it often takes some trial and error. But some tips are below.

Example 5.2 (Parametrizing circles)**.** The implicit equation for the unit circle is $x^2 + y^2 = 1$. Because the functions $\cos t$ and $\sin t$ satsify the equatio

$$\cos^2(t) + \sin^2(t) = 1$$

We see that if $x = \cos(t)$ and $y = \sin(t)$ then (x, y) must lie on the unit circle. Similarly, the equation $f(t) = (r\cos(t), r\sin(t))$ parameterizes a circle of radius r centered at $(0, 0)$, and

$$f(t) = (r\cos(t) + h, r\sin(t) + k)$$

parameterizes a circle of radius r centered at (h, k).

Similar parameterizing an implicit plane curve by finding functions which satisfy the relations, one can parameterize curves that are the intersection of two known surfaces.

Example 5.3. Parameterize the twisted cubic, lying on $y = x^2$ and $z = x^3$. Here if $x = t$ then we know $t = t^2$ and $z = t^3$: this fully specifes a point in 3-dimensional space, so we have our parameterization

$$f(t) = (t, t^2, t^3)$$

Example 5.4. Parameterize the intersection of the cylinder $x^2+y^2=3$ and the plane $x+y+z=1$. On the cylinder, x and y both lie on a circle of radius $\sqrt{3}$: so we can write $x=\sqrt{3}\cos(t)$ and $y=\sqrt{3}\sin(t)$. The cylinder equation doesn't tell us anything about z, so its no help there. But - we can solve the plane for z in terms of x and y to get $z=1-x-y$. Now we can plug in what we know x and y to be to get the parameterization:

$$\gamma(t)=\left(\sqrt{3}\cos(t),\sqrt{3}\sin(t),1-\sqrt{3}\cos(t),\sqrt{3}\sin(t)\right)$$

Example 5.5. Intersection of $4y=x^2+z^2$ and $y=x$. We know already because $y=x$ that our points in space are going to look like (x,x,z). We can substitute this idea into the first equation to see that $4y$ becoes $4x$, and so

$$x^2-4x+z^2=0$$

This is the equation for a circle! We can find its center and radius by completing the square:

$$x^2-4x=x^2-4x+4-4=(x-2x)^2-4$$

So, this is the circle

$$(x-2)^2+z^2=4$$

Which is a circle of radius 2 centered at $(2,0)$. We can parameterize it as $x=2\cos(t)+2$ and $z=2\sin(t)$. So, in 3D along the plane $y=x$ this becomes

$$f(t)=(2\cos(t)+2,2\cos(t)+2,2\sin(t))$$

5.1.1. Same Curve, Different Parameterizations

Different parameterizations can describe the same curve: since a parameterization is like an *animation* of the curve, you can think of this as tracing out the curve at different speeds.

Example 5.6 (Different Parameterizations of the Circle)**.** All three of these parametric curves trace out the unit circle.

$$(\cos t,\sin t)$$

$$(\cos 2t,\sin 2t)$$

$$(\cos t,-\sin t)$$

The first traces it at unit speed, counterclockwise. The second at twice the speed in the same direction. And the third, at unit speed but *backwards* (clockwise).

Example 5.7 (Different Parameterizations of $y^2 = x^3$). We can parameterize the implicit curve $y^2 = x^3$ in several ways: taking the square root of both sides gives y as a function of x (with a plus and minus component), $y = \pm\sqrt{x^3}$ so one possible parameterization is

$$\alpha(t) = (t, \pm\sqrt{t^3})$$

This isn't the nicest, as we have that ± sign. Another thing we could do is take the cube root: this doesn't cause any ± ambiguity, and gives x as a function of y, or $x = \sqrt[3]{y^2}$, leading to the parametric curve

$$\beta(t) = (\sqrt[3]{t^2}, t)$$

A third option is to find a function for $x(t)$ where when we cube it, we get the same thing as if we squared the function we chose for $y(t)$. This is of course tricker - but here one option is to take $x = t^2$ and $y = t^3$. Then $x^3 = t^6$ and $y^2 = t^6$ so $x^3 = y^2$ and our curve is

$$\gamma(t) = (t^2, t^3)$$

5.1.2. New Curves from Old

Once we know a few parametric curves (circles, lines, some implicit curves, etc) - its easy to find more by *modifying* the ones we already know! Some of the simplest such tranfsormations we've already used in the case of circles, scaling and translation.

Theorem 5.1 (Scaling a Parametric Curve). *If $f(t) = (x(t), y(t))$ is a parametric curve, then $rf(t) = (rx(t), ry(t))$ is a curve where all the coordinates are r times a big.*

Theorem 5.2 (Translating a Parametric Curve). *If $f(t) = (x(t), y(t))$ is a parametric curve, then $f(t) + (a, b) = (x(t) + a, y(t) + b)$ is the result of shifting the curve over by (a, b).*

Of course, more interesting transformations are also possible - and it's easiest to see this through a couple examples!

5.2. Case Study: Spirals

We will make and understand a collection of *spirals* starting with the basic equation of the unit circle

$$(\cos(t), \sin(t))$$

Example 5.8 (Archimedean Spiral)**.** The archimedean spiral is the curve that rotates about the origin at unit speed, but after rotating angle t, lies not at unit distane (like a circle) but at distance t from the origin. To parameterize, we *multiply* the circle by t:

$$\gamma(t) = (t\cos(t), t\sin(t))$$

Example 5.9 (Logarithmic Spiral)**.** The logarithmic spiral moves away from the origin *exponentially fast*, instead of linearly. This has radius at time t equal to e^t, so

$$\gamma(t) = (e^t\cos(t), e^t\sin(t))$$

Different functions $r(t)$ for the radius multiplied by the circle give spirals that move outwards (or inwards) at different speeds. Try making some of these in the graphing calculator above!

Exercise 5.1 (Whirlpool)**.** Can you make a spiral that rotates about the origin at unit speed, but whose radius asymptotes to 2, never getting any larger?

Example 5.10 (Helix)**.** A helix is a curve where x, y travel around a circle, and z increases at unit speed. For example, the unit helix is

$$\gamma(t) = (\cos(t), \sin(t), t)$$

Example 5.11 (Slinky-Like Helix)**.** What if we want a helix like curve to move vertically at an uneven rate? Replace the z component with a more interesting function of t! For instance, if $z = e^t$ then the curve bunches up as $t \to -\infty$ along the xy plane:

$$\gamma(t) = (\cos(t), \sin(t), e^t)$$

Example 5.12 (Spiral On a Cone)**.** The surface $z = \sqrt{x^2 + y^2}$ traces out a cone - the height is equal to the radius! How can we draw a spiral on the surface of the cone? Well, if we know what we want the spiral to do in its x and y components, we can calculate the z component using the formual above! For instance, given the archimedean spiral $(t\cos(t), t\sin(t))$ we see $z = \sqrt{(t\cos t)^2 + (t\sin t)^2} = t$. Thus, the curve is

$$\gamma(t) = (t\cos t, t\sin t, t)$$

5.3. Videos

A recap of parametric curves

https://youtu.be/bb4bSCjlFAw?si=3cHJuYIJvD9N6VuD

Parametric curves and elimination of parameters:

https://youtu.be/97pe-QlSGqA?si=pKIz2u9f3an3fPC3

6. Calculus

(Relevant Section of the Textbook: 13.2 Derivatives and Integrals of Vector Functions)

A parametric curve is made of n component functions, which are familiar functions $\mathbb{R} \to \mathbb{R}$ from single variable calculus. Thus, the calculus of curves is simply doing the calculus of single variable functions n times!

6.1. Limits

Taking limits of a parametric curve is just taking the limit of each component function.

Definition 6.1 (Limits of Parametric Curves)**.** If $\vec{r}(t) = (x(t), y(t), z(t))$ is a parametric curve, then limits are computed componentwise:

$$\lim_{t\to a} r(t) = \left(\lim_{t\to a} x(t), \lim_{t\to a} y(t), \lim_{t\to a} z(t)\right)$$

Example 6.1 (Limits of Parametric Curves)**.** Let $\vec{r}(t)$ be the following parametric curve

$$\vec{r}(t) = \left\langle \frac{1}{t+1}, \frac{\sin(t)}{t}, \frac{3t+t^2}{t} \right\rangle$$

Compute the limit $\lim_{t\to 0} \vec{r}(t)$.

Computing the limit componentwise we see we just need to evaluate three limits:

$$\lim_{t\to 0} \frac{1}{t+1}$$

$$\lim_{t\to 0} \frac{\sin(t)}{t}$$

$$\lim_{t\to 0} \frac{3t+t^2}{t}$$

The first of these is continuous at zero so we can just plug in. THe second need L'Hospital's rule, and the third needs us to cancel a t from the numerator and denominator before plugging in, to get

$$\lim_{t\to 0}\vec{r}(t) = \langle 1, 1, 3\rangle$$

6.2. Differentiation

Recall the single variable definition of the derivative:

$$f'(t) = \lim_{h\to 0}\frac{f(t+h)-f(t)}{h}$$

The same definition works for parametric curves:

Definition 6.2 (Differentiating Curves)**.** The derivative of a parametric curve $\vec{r}(t)$ at a point t is given by the following limit:

$$\lim_{h\to 0}\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$

The numerator here is a *vector*, which gets infinitesimally small as $h \to 0$, connecting two closer and closer together points of the curve. We then rescale this vector with scalar multiplication, dividng by h to keep the vector's length from collapsing. In the limit, this converges to a vector $\vec{r}'(t)$ which is *tangent* to the curve.

https://stevejtrettel.site/code/2023/parametric-curve-tangent

However, we don't need to calculate the derivative using this limit statement every time! We can use the fact that limits distribute over the components of the function to prove that we can also take the derivative *one component at a time.*

Theorem 6.1 (Differentiating Curves Componentwise)**.** *If* $\vec{r}(t) = (x(t), y(t), z(t))$ *is a parametric curve, then*

$$\vec{r}'(t) = \langle x'(t), y'(t), z'(t)\rangle$$

Because of this, its straightforward to show that differenatiation of vector functions obeys the familiar laws of single variable calculus: you can break it up over sums, and pull out scalars. But now there are three types of products (do we multiply the vector function by a scalar function, or dot or cross product it with another vector?)

Theorem 6.2 (Differentiation Product Laws)**.**

$$\left(f(t)\vec{r}(t)\right)' = f'(t)\vec{r}(t) + f(t)\vec{r}'(t)$$

$$\left(\vec{c}(t)\cdot\vec{r}(t)\right)' = \vec{c}'(t)\cdot\vec{r}(t) + \vec{c}(t)\cdot\vec{r}'(t)$$

$$\left(\vec{c}(t)\times\vec{r}(t)\right)' = \vec{c}'(t)\times\vec{r}(t) + \vec{c}(t)\times\vec{r}'(t)$$

There is also a chain rule: we can't compose a vector function *inside* another vector function, but we can plug a scalar function in as the parameter in a curve!

Theorem 6.3 (The chain rule).

$$\left(\vec{r}(f(t))\right)' = \vec{r}'(f(t))f'(t)$$

You probably notice a similarity to the single variable calculus versions in all of these: they're as close as possible, except now being about vector functions! But these simple looking rules actually provide us a new powerful set of tools, they tell us about the *rate of change* at the same time as dot and cross products - which we know can measure areas and angles!

As one quick example, we'll prove a very useful fact about curves defined by vectors of constant length.

Theorem 6.4 (Curves on the Sphere). *If a curve $\vec{r}(t)$ never changes in length, so $\|\vec{r}(t)\| = k$ for all time, then $\vec{r}'$ is orthogonal to r.*

Proof. Since the magnitude of a vector can be calculated from its dot product, we see that

$$\vec{r}\cdot\vec{r} = k^2$$

Taking the derivative of this with the product rule, we find

$$\vec{r}'\cdot\vec{r} + \vec{r}\cdot\vec{r}' = 0$$

The dot product is commutative (order doesnt matter) so we can re-arrange the left hand side:

$$2\vec{r}\cdot\vec{r}' = 0$$

But dividng by two - this says that the dot product of $\vec{r}$ and $\vec{r}'$ is zero! So these two vectors are orthogonal. □

6.3. Integration

The story of vector valued integration is similarly straightforward. Recall the definition of integration via Riemann sum:

$$\int_a^b f(x)dx = \lim_{N\to\infty}\sum_{i=1}^N f(x_i)\Delta x$$

We can apply the same definition to vector valued functions, as its composed of just the operations of addition and scalar multiplication!

Definition 6.3 (Integrating Curves)**.** If $\vec{r}$ is a parametric curve, its *vector valued integral* on the interval $[a, b]$ is defined by the following Riemann sum:

$$\int_a^b \vec{r}(t)dt = \lim_{N\to\infty} \sum_{i=1}^{N} \vec{r}(t_i)\Delta t$$

But wait! Both scalar multiplication and addition are things we can do *componentwise* for a vector. So we can break this big Riemann sum up into a standard riemann sum in each component. Taking the limit, this tells us we can integrate vector functions componentwise.

Theorem 6.5 (Integrating Curves Componentwise)**.** *If* $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$*, the vector valued integral of* $\vec{r}$ *on* $[a, b]$ *is given by*

$$\int_a^b \vec{r}(t)dt = \left\langle \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt \right\rangle$$

We will only find limited use of this in our class, as there are other types of integrals along curves that will prove more important. Nonetheless this does show up in many applications of multivariable calculus to engineering and physics, where one may wish to recover position from velocity, or velocity from acceleration.

Example 6.2 (Displacement from Veloicty)**.** If $\vec{v}(t)$ is a parametric curve giving the *velocity* of a particle at time t, then $\int_a^b \vec{v}(t)dt$ is teh *displacement vector* for its net travel between $t = a$ and $t = b$.

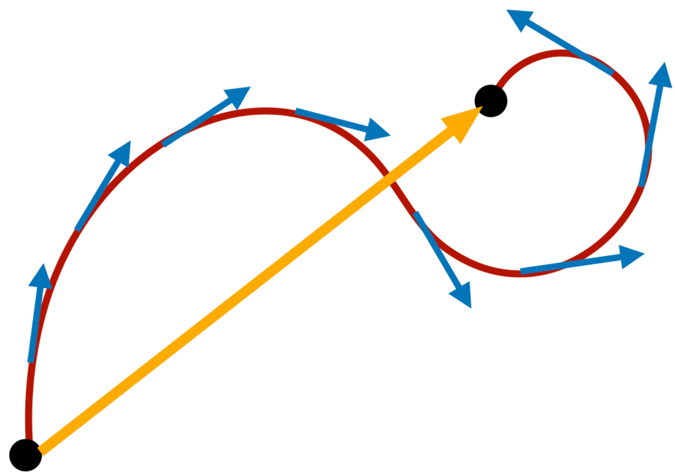


Figure 6.1.: The displacement (yellow vector) can be calculated by integrating the velocity vectors (blue) even if we do not know the parametric curve itself (red).

The same is true for *acceleration*: all of our phones have a sensor inside called a three axis accelerometer. These measure the acceleration as a function of time

$$\vec{a}(t) = \langle a_x(t), a_y(t), a_z(t) \rangle$$

But this is not what software running on the phone wants or needs: it cares about your *position* in space! (Say, if you're using your phone in augmented reality). To get this, it *integrates* the acceleration to get velocity, and then *integrates again* the veloctity to get position! P For any engineers in the class - if you have accelerometers in the Hive: this could be a fun project to code up! Write a small python program to numerically integrate (ie compute a Riemann sum) from the output data of an accelerometer, and track your hand's position as you move it around.

6.4. Videos

Calculus Blue series on calculus of curves:

https://youtu.be/8WzHSgE0Kus?si=D217ouCNQBLxRsI9

Here are some videos practicing the concepts that we have done examples of in lecture. First, limits of vector functions.

https://youtu.be/bhP9cfB90Kc?si=pS7CeIkhktSTijRu

Next, derivatives of vector functions.

https://youtu.be/i9FugTcqWKo?si=p7BoWwVic8-6d-Wu

https://youtu.be/vcwvzUVLPw0?si=mtZpSwCJVJ5Gu5gi

And the proofs of the differentiation laws for the dot product and cross product:

https://youtu.be/fbzEYaYOgfo?si=9aw3IktC0ciLq_Dp

https://youtu.be/vykDXI9OjDM?si=3zDR8TnwSPPSsP9

Also, integrating vector functions.

https://youtu.be/RsGuE5OZqMg?si=onfxxJTYYrVZ5dTu

7. Geometry

(Relevant Section of the Textbook: 13.3 Arc Length and Curvature)

7.1. Arc Length

Definition 7.1 (Infinitesimal Arclength)**.** If $\vec{r}(t)$ is a parametric curve, its infinitesimal arclength is measured by

$$ds = |\vec{r}'(t)|\, dt$$

This makes sense: after all the derivative $\vec{r}'(t)$ is the *velocity*, $\|\vec{r}'(t)\|$ is the *speed*, and dt is an infinitesimal length of time. Thus, the product $\|\vec{r}'(t)\|dt$ is an infinitesimal bit of *distance* - a small length along the curve. To take this infinitesimal information and get something useful out - we need to *integrate* along the curve.

Definition 7.2 (Arclength)**.** If $\vec{r}(t)$ is a parametric curve, its length between $t = a$ and $t = b$ is given by

$$L = \int_a^b ds = \int_a^b |\vec{r}'(t)| dt$$

Example 7.1 (Arclength of a Helix)**.** Find the arclength of $\vec{r}(t) = (\cos(t), \sin(t), t)$ from $t = 0$ to $t = 2\pi$. First, we need to find the velocity $\vec{r}'$:

$$\vec{r}'(t) = \langle -\sin(t), \cos(t), 1\rangle$$

Next, we need to take this velocity and find the speed:

$$\|\vec{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$$

Finding arclenght is just integrating this over the domain:

$$\int_0^{2\pi} \|vecr'(t)\| dt = \int_0^{2\pi} \sqrt{2} dt = 2\pi\sqrt{2}$$

Often arclength integrals can be challenging to do, because of the square root. But with some algebra and integration tricks, alot can be learned.

From this idea, we can define the *arclength function* whcih measures the length of a curve $\vec{r}(t)$ from a starting point $t = a$:

Definition 7.3 (The Arclength Function)**.** If $\vec{r}(t)$ is a parametric curve, for any given starting point $t = a$ we may define the *arclength function* which measures the length of curve between a and t:

$$s(t) = \int_a^t |\vec{r}'(u)| du$$

(Note we have changed the variable of integration so that t is not used in two different contexts)

What is the arclength function for the helix in our earlier example, starting from $t = 0$? Since $\|\vec{r}'(t)\| = \sqrt{2}$, we see that

$$s(t) = \int_0^t \sqrt{2} dt = \sqrt{2} t$$

This tells us that after t seconds, we have traced out $\sqrt{2}t$ units of arclength. How could we reparameterize this curve so that its arclength function is just $s(t) = t$ (tracing out t units of arc in t units of time)?

Definition 7.4 (Unit Speed Curve)**.** A curve $\vec{c}(t)$ is *unit speed* if $\|\vec{c}'(t)\| = 1$ for all times t. This means that it after t seconds, the curve has traversed t units of length. For this reason, we also call unit speed curves *arclength parameterized curves.*

In our example, to make the helix unit speed we need to slow it down by a factor of $\sqrt{2}$: that is, we need $\vec{r}(t/\sqrt{2})$:

$$\vec{r}\left(\frac{t}{\sqrt{2}}\right) = \left(\cos\frac{t}{\sqrt{2}}, \sin\frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right)$$

7.2. Curvature

Besides the length of a curve, one of the most powerful things calculus allows us to do is rigorously study its *curvature.* How can we quantify the fact that some curves bend gently and others turn sharply in space? One means of trying to do this is by looking at the tangent vectors to the curve, and trying to determing how quickly they are changing.

Of course, there's a complication to this: a tangent vector can change in *length* without changing in direction. This doesn't mean that a curve is *curving*, but rather that the particle tracing it out is *accelerating*.

To remove this worry, we define the *unit tangent vector* to a curve. Just divide the derivative by its magnitude!

Definition 7.5 (Unit Tangent Vector)**.** The unit tangent vector to the curve $\vec{r}(t)$ is the vector of length 1 which is parallel to $\vec{r}'(t)$:

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

https://stevejtrettel.site/code/2023/parametric-curve-tangent

This allows a clean definition of curvature: it is how much the unit tangent vector turns per arclength.

Definition 7.6 (Curvature of a Curve)**.** The curvature of a curve is

$$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\vec{T}'(t)\right| \Big/ |\vec{r}'(t)|$$

Where the second equality is derived via the chain rule:

$$\left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}}{dt}\frac{dt}{ds}\right| = \left|\frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}}\right| = \frac{\left|\vec{T}'\right|}{|\vec{r}'|}$$

This formula is difficult to apply in genreral, as the unit tangent vector T might have a pretty scary looking formula, and so taking its derivative can be a lot of work. Doing some calculus we can get a simpler formula:

Theorem 7.1 (Curvature of a Curve)**.** *The curvature of $\vec{r}(t)$ is given by*

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

This is something that's relatively easy to compute (though perhaps tedious) from any parameterization: you just need to find the first and second derivatives, take a cross product, and then plug into the formula!

But, if we further restrict ourselves to the case that $\vec{r}(t) = (t, f(t))$ traces the graph of a function, we can simplify this calculation even more:

Theorem 7.2 (Curvature of a Graph)**.** *If $y = f(x)$ is a function, the curvature of its graph is*

$$\kappa(x) = \frac{|f''(x)|}{|1 + f'(x)|^{3/2}}$$

The below graphing calculator lets you entere a function $\kappa(s)$ that specifies the curvature of a curve, and then it computes a curve which has that curvature! Try even just the case $k(s) = s$ and think about the result - what sort of curve do you expect to see if the curvature grows linearly along the length of the curve?

https://stevejtrettel.site/code/2022/curvature-torsion

7.3. Framing a Curve

The unit tangent vector provides us with a very useful “pointer” - always oriented directly along a curve. But in any serious application of parametric curves, we need more information: we would like a whole x, y, z coordinate frame at each point of the curve.

To start, we’ll look for one vector which is orthogonal to, or *normal to* our curve. How can we find one? Well, the unit tangent is of constant length (its the *unit tangent*, after all). We can use the *product rule for dot products* to understand T':

$$(T\cdot T)=1 \implies (T\cdot T)'=0$$

$$(T\cdot T)'=T'\cdot T+T\cdot T'=2T\cdot T'$$

Thus, we see that $2T\cdot T'=0$, so T is orthogonal to T'!. To find a unit vector orthogonal to T, we just need to normalize T'.

Definition 7.7 (Normal Vector).

$$\vec{N}(t)=\frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

Given these two, its easy to find a third unit vector: just take the cross product of T and N! The result is called the *binormal* as its a second normal vector to the curve.

Definition 7.8 (Binormal Vector).

$$\vec{B}(t)=\vec{T}(t)\times\vec{N}(t)$$

Together these three vectors provide a coordinate system at each point along the curve: T measures distance in the tangent direction to the curve, N measures distance in the direction the curve is bending fastest, and B is orthogonal to both. This collection of vectors is called the *Frenet Frame* and is heavily used in computations in physics, engineering, and computer graphics.

Definition 7.9 (Frenet Frame).

$$\vec{T}(t)=\frac{\vec{r}'(t)}{|\vec{r}'(t)}\qquad \vec{N}(t)=\frac{\vec{T}'(t)}{|\vec{T}'(t)|}\qquad \vec{B}(t)=\vec{T}(t)\times\vec{N}(t)$$

https://stevejtrettel.site/code/2023/frenet-frame

7.4. Videos

Here are some useful videos reviewing the sort of examples that we have covered in class:

7.4.0.1. Arclength of Curves

https://youtu.be/TAQPEP9pEhw?si=EIy61FdfPevHZnR0

https://youtu.be/TZJ2btou8_c?si=XjVF1w0CfbcgxXKC

https://youtu.be/X3AAltBz0wo?si=mcKCWeQMpjjz5aO0

Parameterizing a curve with respect to arclength (a unit speed parameterization)

https://www.youtube.com/watch?v=O3nnibgLCCc

7.4.0.2. Unit Tangents and Normals

https://youtu.be/mDUpR1_Qn70?si=RSlHcXX9-1elKMOr

The calculus Blue series on Tangent, Normal and Curvature:

https://youtu.be/qfE2nTaLxD8?si=zbJ_cAFGtVTYEu6z

The binormal vector and the Frenet Frame:

https://youtu.be/VkqTYPq8dX4?si=c0t_GzVKFVwF3dn8

7.4.0.3. Curvature

https://youtu.be/NlcvU67YWpQ?si=ImoRnmwFO0ALQC1R

An application of this: finding the point on a curve where it is maximally curved (say, you wanted to find the sharpest bend of a roller coaster)

https://youtu.be/qpbFlhDbKwI?si=8IhMtdnVH7kZ_MAh

Part III.

Differentiation

8. Graphs & Level Sets

So far in this course we have been studying parametric curves. These are functions which have a *single input* (the parameter) and multiple outputs (perhaps x, y, z or a color for each pixel of an image, etc.) Now we turn to reverse the situation, a function which has *multiple inputs* but a *single output*! These are sometimes called multivariable functions (to emphasize the number of inputs), but also are called scalar functions (to emphasize the output is a single number, or a *scalar*).

In physics, functions with multiple inputs are often called fields. So yet another name for this class of things is a *scalar field*! Just as I have done in these notes elsewhere (with vector notation, for example) I will try to use all of these terms to help you prepare for the real world where there's no standardization.

Definition 8.1 (Multivariate Function)**.** A function $f : \mathbb{R}^n \to \mathbb{R}$ taking in an n-tuple of real numbers, and outputting a single real number.

uch a function is written $f(x, y, z, \ldots)$ for example

$$f(x, y, z) = 2x^2 - 3x + sin(e^z)$$

The **domain** of a scalar function is the set of inputs for which the function makes sense.

Example 8.1. The domain of $f(x, y) = \ln(x^2 + y^2 - 1)$ is the *exterior* of the unit circle in the plane $D = \{(x, y) \mid x^2 + y^2 > 1\}$. This is because the logarithm of zero or a negative nuumber is underfined, so the only allowable inputs are when $x^2 + y^2 - 1$ is positive.

Our goal in this chapter is to get comfortable with multivariable functions from two different perspectives: drawing graphs and drawing level sets.

8.0.1. Examples

Scalar fields show up everywhere in mathematics and the sciences. Consider the temperature in a room: this is a function that takes in a point (say, with three coordinates (x, y, z)) and returns a single number - the temperature of the air at that point in space. This is then a function $T : \mathbb{R}^3 \to \mathbb{R}$.

If we wanted to track the temperature over time, this could be done with a function $\mathbb{R}^4 \to \mathbb{R}$ which takes (x, y, z, t) and returns the temperature there.

Of course - temperature is just an arbitrary (but conceptually useful!) example. **Any** quantity you can measure at differet points in space(time) gives a scalar function of 3 or 4 variables!

Exercise 8.1. Come up with some examples of scalar functions that you think about in daily life (without necessarily having ever thought about them mathematically!)

8.1. Graphs

The *graph* of a 1-variable function f is a curve in the plane: its the set of points (x, y) where $y = f(x)$. We can make a similar definition for the *graph* of a scalar function

Definition 8.2. The graph of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a subset of $\mathbb{R}^{n+1}$: its the set of points $(\vec{x}, y)$ where $\vec{x} \in \mathbb{R}^n$ and $y = f(\vec{x})$.

This isn't that useful for *visualizing* unless dimenisons are pretty small! If there are n inputs and 1 output we need $n + 1$ dimensions to put them all on a graph. And, if we want to actually see the graph it better fit inside of 3D space so n can only be 1 (the case we already know!) or 2.

If $n = 2$, then we have two input variables and its easiest to think of them as being on the xy plane. Then we may name the output variable z, and plot our function $z = f(x, y)$ as the points $(x, y, f(x, y))$ in 3D space.

Here's a graphing calculator for functions $z = f(x, y)$:

https://stevejtrettel.site/code/2023/graph3d

8.1.1. How to Draw Graphs

Drawing a 3D graph is difficult to do in general: but often we can use our knowledge of 2D graphs to try and build up an understanding by **slicing**. The idea is to take a function $z = f(x, y)$ and plug in *constant* values for one of the variables, then try to stack these slices to get a model of the entire surface.

Example 8.2 (Slicing $z = xy$)**.** Draw the slices of $z = xy$ - When $y = 0$ this is the horizongal line $z = 0$ - When $y = 1$ this is the line $z = x$ - When $y = 2$ this is the line $z = 2x$

Thus, as y increases, the *slope* of the graph's cross section increases!

Example 8.3 (Slicing $z = ye^x$)**.** Here the slices for $y = 0$, $y = 1$ and $y = 2$ look like 0 e^x and $2e^x$. They are copies of the exponential getting steeper and steeper.

Example 8.4 (Slicing $z = x^2 + y^2$)**.** Slices of this function are parabolas in both the x and y directions. Fixing y equal to different values, the parabolas shift upwards! For $y = 0, 1, 2$ we get

$$z = x^2, \qquad z = x^2 + 1 \qquad z = x^2 + 4$$

8.1.2. Useful Graphs to Know

There are a couple multivariable functions whose graphs are good to know. Indeed - we've already met some of these before! If $f(x, y)$ is a *linear* equation like $ax+by+c$, the graph is the set of points

$$z = ax + by + c$$

This is a plane! (It might help to rewrite as $ax + by - z = -c$ to see its an implicit equation, with normal vector $\langle a, b, -1\rangle$.)

What about $z = x^2 + y^2$? We saw this one above in our discussion of slicing, and we also saw it earlier (in "Shapes"): its a parabola of revolution!

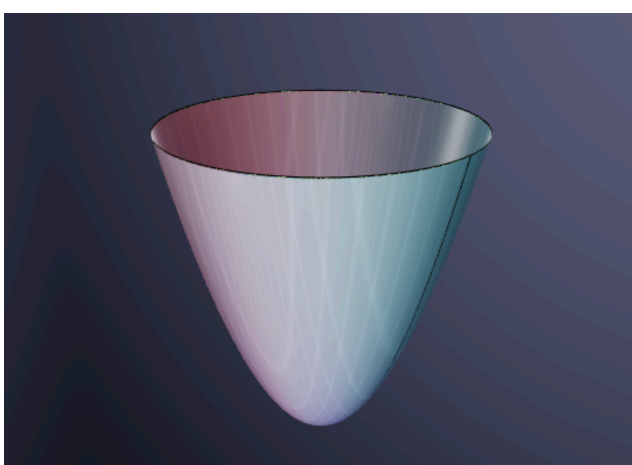


Figure 8.1.: The graph of $z = x^2 + y^2$

Another useful function to know is the *saddle surface* $z = x^2 - y^2$. We also met this one back in the discussion of "Shapes".

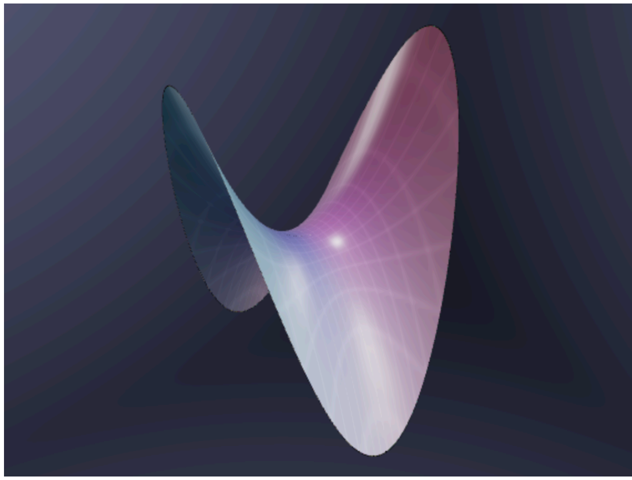


Figure 8.2.: The graph of $z = x^2 - y^2$

8.2. Level Sets

Above we looked at one means of drawing a graph by slicing: we attempted to slice it by *vertical planes* into the graphs of simpler 1-dimensional functions that we already knew! This works sometimes, but if you can't quickly stack the slices into a coherent image in your mind, knowing the slices won't help you with much else.

So here, we seek other methods of understanding these functions, beyond their graphs. By far the most useful way to depict multivariable functions is by instead slicing with *horizontal planes* and drawing their level sets.

Definition 8.3 (Level set)**.** The level set corresponding to $c \in \mathbb{R}$ for a function $f(x, y)$ is the set of points in the plane that f maps to c:

$$L_c = \{(x, y) \mid f(x, y) = c\}$$

Here's a graphing calculator that will draw for you the level sets of a function: I often think about "sea level" when I think of a level set - a coastline is the level set $h(x, y) = 0$ above the water. And different level sets correspond to what the coastlines *would be* if the sea was different heights.

https://stevejtrettel.site/code/2023/contour-slicing

A *contour plot* is a drawing of the *domain* of a function, with level sets representing various values of the *range*. These are perhaps most familiar from elevation maps. Drawing multiple level sets at once gives a good sense of the behavior of the entire function, though its most effective when the individual level sets are *labled* somehow (often by color) so you can get a sense of their relative values.

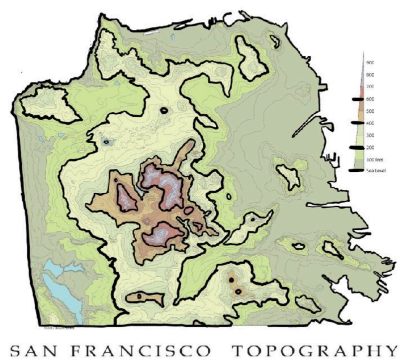


Figure 8.3.: A topographic map of San Francisco

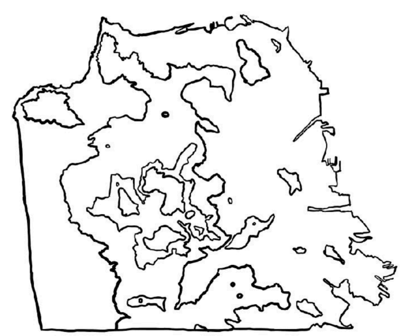


Figure 8.4.: Some Level curves of the map. Can you see lone mountain?

Below, we draw the same map but only plot the level curve corresponding to 200ft above sea level.

$$L_{200} = \{(x, y) \mid h(x, y) = 200\}$$

Figure 8.5.: The 200ft-level set

A single level set of a function is actually something that you've encountered in previous courses: we called it an *implicit equation* as it defines a shape implicitly by saying (x, y) lives on the curve if $h(x, y)$ has a specific value - instead of specifying it *explicitly* liek a function.

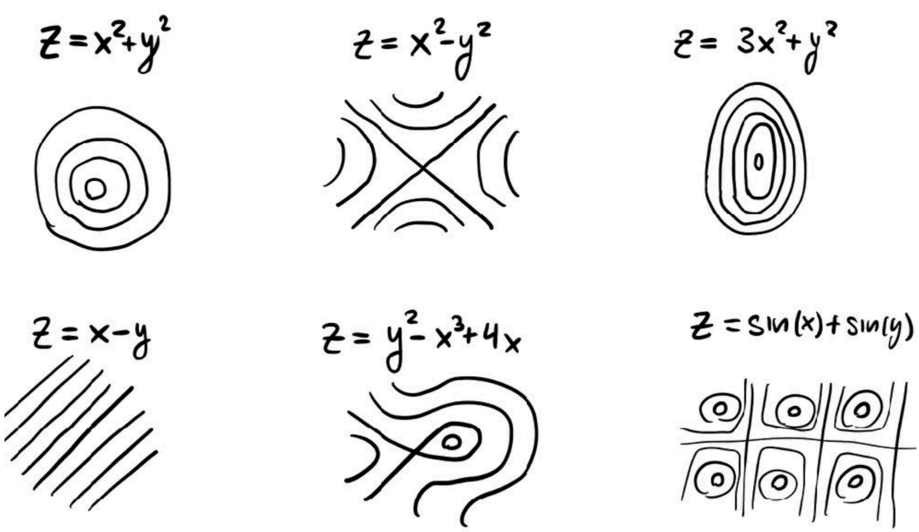


Figure 8.6.: Some contour plots of various functions.

Below is a graphing calculator to let you draw some contour plots. Can you imagine the 3D shape from its 2D slices?

https://stevejtrettel.site/code/2023/levelsets

After looking at several functions and their level sets, you'll start to notice that there are a couple "important behaviors" that show up again and again. These are

- Concentric rings, around a point
- Nearly parallel lines
- Two lines crossing each other.

These signify three important types of behaivor, which we can see by looking back to our "useful graphs to know"

Concentric rings around a point signifies the function has either a maximum or a minimum there.

Nearly parallel lines means the function is increasing or decreasing there.

Two lines crossing means that we are at a *saddle shaped point* on our graph - it increases in two directions and decreases in the other two.

It turns out, that all behavior of level sets is built out of these three basic behaivors.

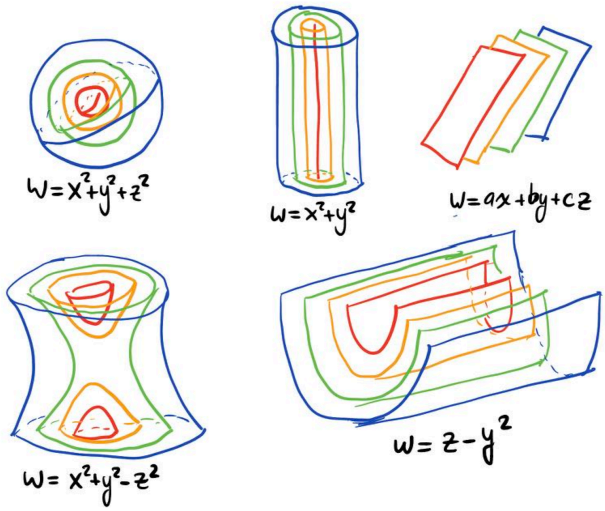
8.3. Functions of ≥ 3 Variables

Drawing a contour plot is a form of *dimension reduction*: we've managed to understand the behavior of a function f whose graph $z = f(x, y)$ lies in 3 dimensional space, by only looking at a 2-dimenisonal image (its domain, covered in level sets).

This technique can help us level up our intuition to functions of *three variables*: things like $w = f(x, y, z)$ whose graphs would naturally live in four dimensional space!

Exercise 8.2. What do the level sets of the function $w = x^2 + y^2 + z^2$ look like, for different values of w?

In three dimensions there are more types of *basic behavior* than the ones we saw in 2D. You don't need to learn all of them: but to try and get some intuition for the fourth dimension its a good exercise to try and imagine what the graphs of these functions must be like, if their contours are drawn below.



Three dimensional level sets describe *implicit surfaces* which are extremely useful objects. As we've already seen with curves, sometimes writing down a paramterization can be hard. Adn this is even more true for surfaces! So having another way to express complicated ones can be a huge help. Below are two examples where I have used implicit surfaces in mathematical rendering.

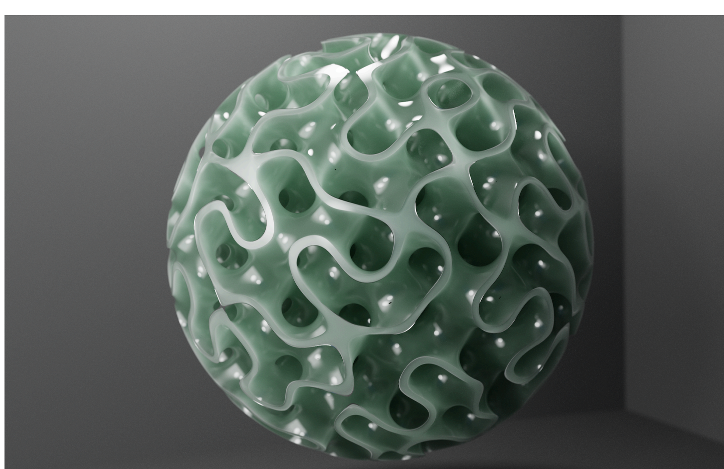


Figure 8.7.: The implicit surface $\sin(x)\cos(t) + \sin(y)\cos(z) + \sin(z)\cos(x) = 0$. This is a good approximation to a particularly important mathematical surface called the *gyroid*.

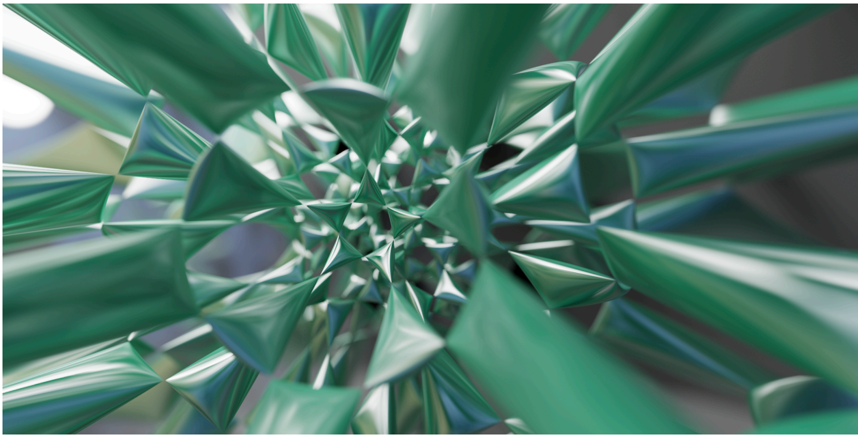


Figure 8.8.: This surface is called the *Barth Decic* and arises in a field of mathematics called algebraic geometry. Its equation is given below

$$
\begin{aligned}
&8(x^2 - p^4 y2^)(y^2 - p^4 z^2)(z^2 - p^4 x^2)\left(x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2\right) \\
&+ a(3 + 5p)\left((x^2 + y^2 + z^2 - a)\right)^2 \left((x^2 + y^2 + z^2 - (2 - p)a)\right)^2 \\
&= 0
\end{aligned}
$$

8.4. Videos

The Calculus Blue Series introduction to multivariate functions:

https://youtu.be/e2TJpv39SJo?si=mGoSRbZ7BOqstCZi

https://youtu.be/gbj0YfdovxU?si=7t8M9qKPA8JkPp4U

The Calculus Blue series on Level Sets:

https://youtu.be/zEvGUXXW1BI?si=54UKw1EULxEsU6lV

9. Partial Derivatives

How do we differentiate a function with multiple inputs? We will learn several ways to do this throughout the course. But all methods rely on one fundamental idea: the *partial derivative.*

9.1. Geometry of Partial Derivatives

If we take a function $f(x, y)$ and hold the y value constant, we get a function of just x. For example, if $f(x, y) = \sin(x * y)$ and $y = 2$ we get the function $f(x, 2) = \sin(2x)$. This sort of function we already know how to take the derivative of!

$$f(x,2)' = \frac{d}{dx} f(x,2) = 2\cos(2x)$$

.

What does this derivative *mean*? Well, we are measuring the slope *in the x-direction* along the line where $y = 2$. Here's a graphing calculator showing this, for different slices and different points!

https://stevejtrettel.site/code/2023/partial-derivatives

What happens if we instead looked at the slice where $y = 7$ Then we'd have $f(x, 7) = \sin(7x)$ and the derivative would be

$$f(x,7)' = \frac{d}{dx} f(x,7) = 7\cos(7x)$$

Thus, whaever y is we see it comes out front as a coefficient via the chain rule. This tells us that if we just call the constant y (and don't bother to specify its numerical value) we should get

$$f(x,y)' = \frac{d}{dx} f(x,y) = y\cos(xy)$$

The only thing confusing here is that unless we know what we are doing, it's hard to tell what the prime means. So we should probably not use this notation when there's more than one variable.

In fact, to signify that we are taking the derivative of a *multivariable function*, its customary to write the d a little fancy as well, using the italic ∂

Definition 9.1 (x-Partial Derivative)**.** If $f(x, y, z, \ldots)$ is a function of multiple variables, the *partial derivative with respect to x is the result of treating all other variables as constants, and differentiating with respect to x. It's denoted

$$\frac{\partial f}{\partial x} = \lim_{h \to 0} \frac{f(x+h, y, z, \ldots) - f(x, y, z, \ldots)}{h}$$

Partial derivatives are ubiquitous in the sciences, and because they are so widely used in so many fields, there are also many common notations for them. I will use many of the notations interchangably in class, to prepare you for the real world: and for reference the most common ones appear below.

Definition 9.2 (Notations for Partial Differentiation)**.** The partial derivative of f with respect to x may be written as

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = \partial_x f = f_x$$

The last notation takes some getting used to at first: *the subscript represents differentiation!* But because of its conciseness, it is very commonly used when performing calculations.

In single variable calculus there was just a single first derivative: f'. Now, the number of first derivatives depends on the number of variables - $f(x, y)$ has two first derivatives f_x and f_y, whereas $g(u, v, w)$ has three! It turns out that it is very convenient to package all of this information together into a single vector, called the *gradient*.

Definition 9.3 (The Gradient Vector)**.** Given a multivariable function $f(x_1, \ldots, x_n)$, its *gradient* is the vector of all first partial derivatives

$$\nabla f = \langle \partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f \rangle$$

The symbol ∇ used in the notation of the gradient will become quite commonplace throughout vector calculus, though this is our first encounter with it. Alone, its pronounced *nabla*, or *del*. Right now, its alright to just think of the gradient a convenient bookkeeping device storing all of the partial derivatives in one handy place. But soon we will see that its *direction* and *magnitude* are actually quite meaningful to the geometry of f, and because of this, the gradient vector lies at the heart of many modern optimization techniques in Machine Learning.

Anything you are used to doing in single-variable calculus can be done with partial derivatives. For example, we can use *implicit differentiation* to take the derivative of implicit equations with multiple variables.

Example 9.1 (Implicit Partial Differentiation)**.** Find the derivative $\partial_x z$ of the expression

$$x^3 + y^3 + z^3 + 6xyz = 7$$

Here we act as though z is implicity a function of x, and we differentiate the whole equation:

$$\frac{\partial}{\partial x}(x^3 + y^3 + z^3 + 6xyz) = \frac{\partial}{\partial x}7$$

Computing this (where we need the product rule on the last term, since we have both an x and a z-where z is implicitly a function of x!) gives

$$3x^2 + 0 + 3z^2\frac{\partial z}{\partial x} + 6yz + 6xy\frac{\partial z}{\partial x} = 0$$

Then, we just solve for $\partial_x z$:

$$(3z^2 + 6xy)\frac{\partial z}{\partial x} = -3x^2 - 6yz$$

$$\frac{\partial z}{\partial x} = \frac{-3x^2 - 6yz}{3z^2 + 6xy}$$

What is implicit differentiation *measuring*? Think back to the 2-dimensional case, from calculus 1. There we had an implicit equation - what we now know to be a *level set* - and we were trying to measure dy/dx: that is, for a small change in x, how much does y have to change, to stay on the level set?

The same picture works here, but now in higher dimension. An implicit equation in x, y, z determines a surface in $\mathbb{R}^3$ - which is the level set of some 3−variable function. And a quantity like $\partial z/\partial x$ tells us how much z has to change to stay on the surface, if we change x a little bit.

9.2. Higher Derivatives

Higher order partial derivatives are no more difficult: each time you take the derivative, you just treat all other variables as constants.

For instance, the second partial x derivative is just what you get by taking the x derivative twice:

$$\partial_x\partial_x(\cos(xy)) = \partial_x(-y\sin(xy)) = -y^2\cos(xy)$$

But you can also take partials with respect to different variables.

$$\partial_x \partial_y (x^3 y^2) = \partial_x (2x^3 y) = 6x^2 y$$

Definition 9.4. A *higher partial derivative* is just the result of taking the partial derivative more than once (perhaps with respect to different variables). When doing this, one needs to be careful with notation: the "derivative notations" are all read like function composition

$$\partial_x \partial_y \partial_z f = \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} f$$

both mean *do the z partial, then the y partial, then the x partial.*

The *subscript notation* is read from inside out:

$$f_{zyx} = ((f_z)_y)_x$$

is the equivalent to the above: doing z first, then diffrentiating with respect to y, and finally with respect ot x.

Theorem 9.1 (Equality of Mixed Partials)**.** *So long as the partial derivatives are defined and continuous, the order in which you take them does not matter.*
$\partial_x \partial_y f = \partial_y \partial_x f$*\$ This works with higher order derivatives as well*

$$f_{xyxx} = f_{xxxy} = f_{yxxx} = \cdots$$

In fact, the second order partial derivatives together form an operation called the *laplacian* which arises in many applications:

Definition 9.5. The Laplacian operator is the sum of the (non-mixed) second order partial derivatives: it is sometimes written as Δ and sometimes as ∇^2: in the plane this is

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x} + \frac{\partial^2}{\partial y}$$

and in higher dimensions, its analogous just with more variables. To take the laplacian of a function, you just find its non-mixed second partials, and add them all up:

$$\Delta f = f_{xx} + f_{yy}$$

One way to imagine what the laplacian is measuring is a sort of *average concavity*: it adds up the concavity in both the x and y directions. Thus, a function like $x^2 + y^2$ has laplacian

$$\Delta(x^2 + y^2) = 2 + 2 = 4$$

so its concave up on average, $-(x^2 + y^2)$ has laplacian -4 so its concave down, and $x^2 - y^2$ has laplacian equal to zero: it is concave up in one direction and concave down in the other: so added together they cancel. Functions whose Laplacian are zero are called *harmonic functions* and play a huge role in understanding differential equations, physics, and engineering.

Just like we packaged all of the first partial derivatives together into one nice object, the Gradient, we do the same with the second partials:

Definition 9.6 (The Hessian (Matrix of 2nd Derivatives))**.** Given a twice differentiable function $f(x, y)$ its *Hessian Matrix* is the 2×2 array of all second derivatives

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

9.3. Partial Differential Equations

Partial derivatives are the language in which much of modern science is written. We saw in the last portion of the course that *vector valued differential equations* are the right language to describe the motion of *single particles*: but what about quantites that depend on more than one variable?

A first example is simply waves on a string: when a guitar string is pulled taught, if you try to pluck it away from rest it *pulls back on you* - the farther you pull it away, the harder it pulls back.

The amount a string curves away from its straight line equalibrium is captured (roughly) by its concavity. And so one simple model of string motion would say the bigger the concavity the faster it wants to "snap back". Said more precisely

The acceleration of the string is proportional to its concavity

Writing this in math - if the string's displacement at position x and time t is given by the function $W(x, t)$, we have the partial differential equation below:

$$\partial_{xx} W = \partial_{tt} W$$

In two dimensions, a wave equation measures the displacement of a circular membrane, like a drumhead or the interior speaker of an earbud. Here we have to account for displacements in both the x and y directions. Thus, the 2 dimensional wave equation is below (now written in the more 'verbose' notation for partial derivatives)

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = \frac{\partial^2 W}{\partial t^2}$$

Here's some solutions to this equation:

https://stevejtrettel.site/code/2022/circular-drum

This same wave equation in three dimensions describes the propagation of *electromagnetic waves* - or light! This was a triumph of 19th century physics, where James Clerk Maxwell derived this wave equation from his equations for the electromangetic field. Below, you can see a (numerically computed) solution to this equation, showing a light beam being focused by a glass lens.

https://stevejtrettel.site/code/2021/wave-eqn-flash

Similar partial differential equations occur throughout physics and engineering. (If you have yet to be convinced of the wide applicability of partial derivatives, look up "continuum mechanics" and try to find a topic you're interested in.)

One final example I'll mention here is quantum mechanics, where the fundamental equation (called the Schodinger equation) is a replacement of Newtons Law (a vector valued differential equation) with a partial differential equation. This big change in the mathematics is what causes people to say that quantum particles can be "like waves".

Below is a calculator I wrote for playing around with the "double slit experiment" in quantum mechanics. This will not come up further in our course, but feel free to ask me if you are interested!

https://stevejtrettel.site/code/2021/schrodinger-double-slit

9.4. Videos

9.4.1. Calculus Blue

https://youtu.be/78bXzF_J3RM?si=AdXYWkEEwH0tkSOX

https://youtu.be/V-_WuybYkyg?si=9hC-AnNP52ehPfcZ

https://youtu.be/T_7FXwImddU?si=ex91sAxbEjLqWZJC

https://youtu.be/DYrs_cOg0E8?si=V3bmWBMgCQrFhuHs

9.4.2. Khan Academy

https://youtu.be/AXqhWeUEtQU?si=XRKrVYhjZC_qWp-p

https://youtu.be/dfvnCHqzK54?si=rasi8aG3Gxt0LFlM

https://youtu.be/J08-L2buigM?si=tNUmEbz-7pFXO2N-

9.4.3. Example Problems:

https://youtu.be/btcSjC5z7WQ?si=SnKosx3_t96cjwi3

https://youtu.be/3itjTS2Y9oE?si=7rWWzXWUHrpllUUC

https://youtu.be/EoEV5-_mLeM?si=LkKZqesDmHVoZqcy

10. Linearization & Approximation

10.1. The Fundamental Strategy of Calculus

Take a complicated function, zoom in, replace it with something linear. Thus we are looking to replace graphs by planes. To start, look back on the tangent line formula from single variable calculus:

$$y = y_0 + f'(x_0)(x - x_0)$$

Our extension to multiple dimensions is just....to add more variables! We need to adjust z not just for changes in x any more, but also for changes in any input.

Theorem 10.1 (Tangent Plane)**.**

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Because the tangent plane is the best linear approximation *to a function at a point, it is often called the* Linearization

Same formula works in higher dimensions, by just adding more terms. Since this gives us an implicit plane, we can re-arrange this to the "standard form" and find the normal vector to the graph

Theorem 10.2 (Normal Vector to a Graph)**.** *At the point* (x_0, y_0) *the plane*

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - z = -z_0$$

is tangent to the graph of f*. Thus, the normal vector is the coefficient vector*

$$n = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1\rangle$$

Note that any scalar multiple of this vector is *also* a normal vector to the graph - this just provides *one such vector.* And, since the z is downwards, this is the *downwards pointing normal vector.*

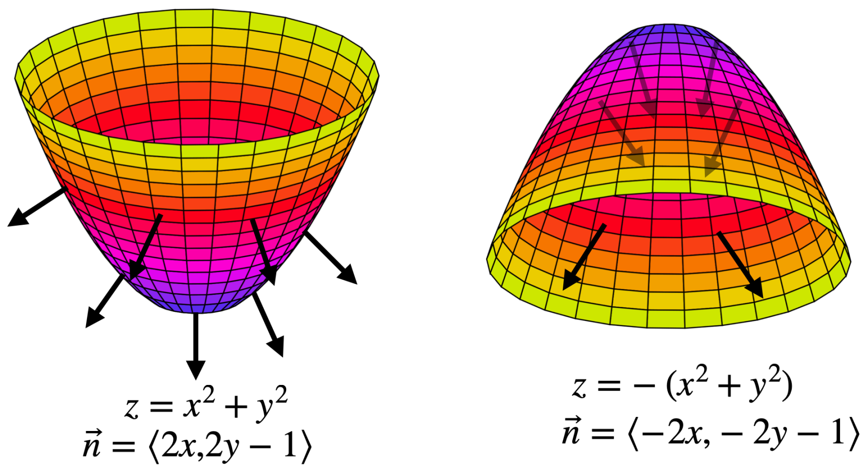


Figure 10.1.: The downward facing normal, on two graphs.

Depending on the application, sometimes we want the *upward facing normal*: that's what you'd get by multiplying this by -1 so that the last coordinate is a 1:

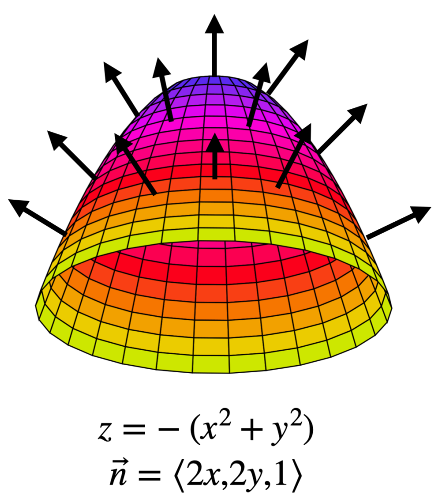


Figure 10.2.: The upward facing normal.

We will use normal vectors to surfaces a lot in the last portion of this course, on Vector Analysis. Here we will often need to be careful, and thinking about whether we want the normal that is pointed up or down in a given application.

Example 10.1. Find the tangent plane to $z = x^2 + 2y^2$ above the point $(x, y) = (2, 3)$.

The differentiability of a function in multiple variables is defined in terms of the existence of a tangent plane: we say that a function $f(x, y)$ is differentiable at a point p if there exists some tangent plane that well-approximates it at that point. Functions we will see are mostly differentiable, but warning there are functions that are not. Luckily, there's an easy way to check using partial derivatives:

Theorem 10.3 (Multivariate Differentiability). *A multivariate function is differentiable at a point, if and only if all of its partial derivatives exist and are continuous at that point.*

10.2. Differentials

The Fundamental Strategy of calculus is to take a complicated nonlinear object (like a function that you encounter in some real-world problem) and zoom in until it looks linear. Here, this zooming in process is realized by finding the tangent plane. Close to the point (x_0, y_0) the graph of the function $z = f(x, y)$ looks like

$$L(x, y) = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Where $z_0 = f(x_0, y_0)$. This is just rehashing our definition of the tangent plane of course: but one *use* for it is to be able to approximate the value of $f(x, y)$ if you know the value of f at a nearby point (x_0, y_0) and also its partial derivatives there.

Example 10.2. Find approximate value value of $x^2 + 3xy - y^2$ at the point $(2.05, 2.96)$.

Using linearization to estimate changes in a value: fundamental to physics and engineering. In 1-dimension, we define a variable called dx that we think of as measuring small changes in the input variable, and $dy = y - y_0$ which measures small changes in the output. These are related by

$$dy = f'(x)dx$$

So, any change in the input is multiplied by the derivative to give a change in the output. We can do a similar thing in more variables. For a function $f(x, y)$, we have the tangent plane above

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Subtracting z_0 and setting $dz = z - z_0$, and similarly for dx, dy we can rewrite this as below:

Definition 10.1 (Differentials)**.**

$$dz = f_x(x,y)dx + f_y(x,y)dy$$

This allows us to easily *estimate* how much z could change if we know how much x and y can change. This is of fundamental importance in *error analysis*, the foundation of all experimental science's ability to compare with theoretical predictions.

Example 10.3. The volume of a cone is given by $V = \pi r^2 h/3$. We have a cone which we measure the height to be 10cm and the radius to be 25cm, but our measuring device can have an error up to 1mm or $0.1cm$. What is the estimated maximal error in volume our measurement could have?

This measurement can also be interpreted *geometrically*: this is the approximate volume of a thin-shelled cone of thickness $1mm$ with radius 25cm and height 10cm.

Example 10.4. The dimensions of a rectangular box are measured to be 75cm, 60cm and 40cm. Each measurement is correct to within $0.05cm$. What is the maximal error in volume measurement we might expect?

10.3. Quadratic Approximations

We've already gotten a ton of use out of *linear approximations* to a multivariable function. But we can learn even more by proceeding to higher derivatives. Here we study the *quadratic approximation* that includes all the first *and* second derivative information. Like in the linear case, the best way to get started is to recall what happens in one variable for the second order term in a taylor series:

$$f(x) \cong f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2$$

The move to multiple variables for the linearization required us just tacking on analogous terms for all additional variables. Happily the same holds true here!

Definition 10.2 (Quadratic Approximation)**.** If $f(x,y)$ is a differentiable function of two variables, its quadratic approximation at a point (a,b) is

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \tag{10.1}$$

$$+\frac{1}{2}\left(f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + f_{yx}(a,b)(y-a)(x-b) + f_{yy}(a,b)(y-b)^2\right) \tag{10.2}$$

Because we know the order in which we take partials doesn't matter, $f_{xy} = f_{yx}$ and so this simplifies to

$$f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \tag{10.3}$$
$$+\frac{1}{2}\left(f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right) \tag{10.4}$$

If you have taken Linear Algebra (or have seen *Matrix Multiplication* before elsewhere) there is a nice way to remember this formula for $\vec{x} = (x,y)$ and $\vec{p} = (a,b)$, writing it out in terms of row vectors, column vectors and matrices, where ∇f is the gradient (vector of first derivatives) and Hf is the Hessian (matrix of second derivatives)

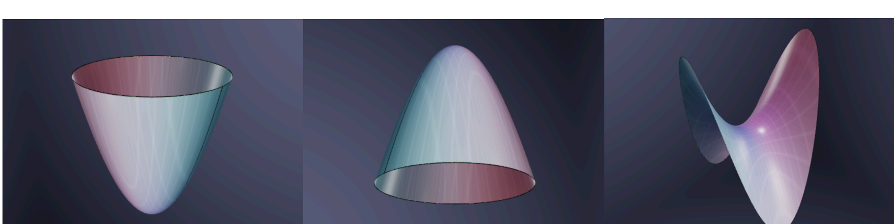
$$f(x,y) \approx f(p) + \nabla f(p)\cdot(\vec{x}-\vec{p}) + \frac{1}{2}(\vec{x}-\vec{p})\cdot Hf(p)(\vec{x}-\vec{p})$$

This formula looks intimidating in any of these forms when written out completely. But its actually super easy to remember!

- Start with the value $f(a,b)$ at the point we know
- Add in first derivatives $f_x(a,b)$ and $f_y(a,b)$ multiplied by how far we've moved in that direction.
- Add in $\frac{1}{2}$ each second derivative, multiplied by the differences $(x-a)(x-a)$, or $(x-a)(y-b)$ or $(y-b)(y-b)$ depending on which second derivatives were taken.

10.3.1. The Geometry of Quadratic Approximations

Once we have a quadratic approximation to a surface we have an even better understanding of what it looks like near a point. Of course, that requires that we *know what quadratic surfaces look like* - and hence why we spent time on those earlier this semester! Generic quadratic surfaces come in three types: hills, bowls and saddles



So computing the quadratic approximation to a function tells us if near that point, a portion of the surface is best approximated by a hill bowl or saddle. We can tell which by computing the quadratic approximation and taking some slices: if the slices are parabolas both opening in the same direction its either a hill or bowl, but if they open in opposite directions its a saddle.

Find a quadratic approximation to $e^x y^2$ at $(1, 2)$. Is this approximation a hill, bowl or saddle?

Note that just because a hill is the best approximation near a point *does not mean we are at a maximum* - we could just be on the side of the hill! But these tools do turn out to be extremely useful for finding maxes and mins: these can occur when the first derivatives are zero. The whole next section of the course notes is devoted to some examples of using the quadratic approximation to do exactly this.

To get a better sense of what's going on, here's a program that draws the quadratic approximation automatically for you.

PROGRAM

Examples of finding quadratic approximations to a function.

10.4. Videos

10.4.1. Calculus Blue:

https://youtu.be/4tNMHtMyDns?si=ITtSJE1NzBJ9bfMe

https://youtu.be/0vma0sbpSqU?si=07L_BNXg7pXjR5A8

https://youtu.be/cK9fHRtv2Rk?si=16JIFRyPtoP3pN0V

10.4.2. Khan Academy

https://youtu.be/o7_zS7Bx2VA?si=pj3GE8x_iiOTBHRH

10.4.3. Example Problems:

https://youtu.be/oJ_LA1AYUl8?si=UmgBmFOqecnmfub7

https://youtu.be/cRGoTL00Ksg?si=0RWUC0TaeBVtuEWd

10.4.4. Quadratic Approximation

https://youtu.be/80bJA_tSbo4?si=lfivM8232oU0o9yn

https://youtu.be/UV5yj5A3QIM?si=nXYejMsdH29EDd40

https://youtu.be/szHMvVXxp-g?si=nZ_CPFqqzKYZxNma

https://youtu.be/fW3snxnCPEY?si=eXRWov85IyXWpspd

11. Extrema

We've developed some powerful tools for working with multivariable functions: we can take their partial derivatives, directional derivatives, and understand the relationship between the gradient and their level sets. Our next goal is to put this knowledge to work and learn how to find maximal and minimal values. This is a critical skill in real world applications, where we are looking to *maximize* efficiency, or *minimize* cost.

Definition 11.1 (Local Extrema)**.** An extremum is a catch-all word for a maximum or a minimum (its an *extreme value*, meaning either largest or smallest). A *local minimum* is a occurs at a point $p = (a, b)$ if the value of the function $f(x, y)$ is always greater than or equal to $f(a, b)$, when x is near a and y is near b. Analogously, a *local maximum* occurs at (a, b) if $f(x, y) \leq f(a, b)$ for all (x, y) near (a, b).

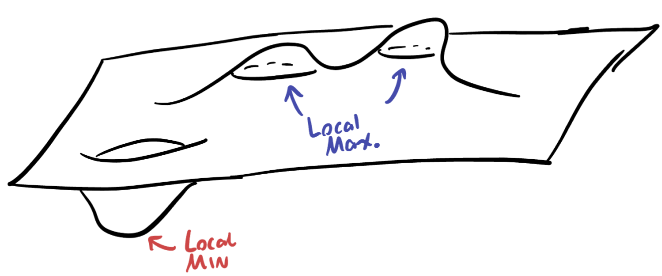


Figure 11.1.: Local Maxima and Minima of a function

In this section we will mainly be concerned with how to find local maxima and minima, though in optimization we will often be after the *maximal value* or *minimal value* of the function overall. These *global maxima or minima* are often just the largest or smallest of the local extrema, so our first step will be to find the local counterparts, then sort through them.

How can we find an equation to specify local extrema? In calculus I we had a nice approach using differentiation: at a local max or min a function is neither increasing nor decreasing so its derivative is zero. The same technique works here, where we consider each partial derivative independently!

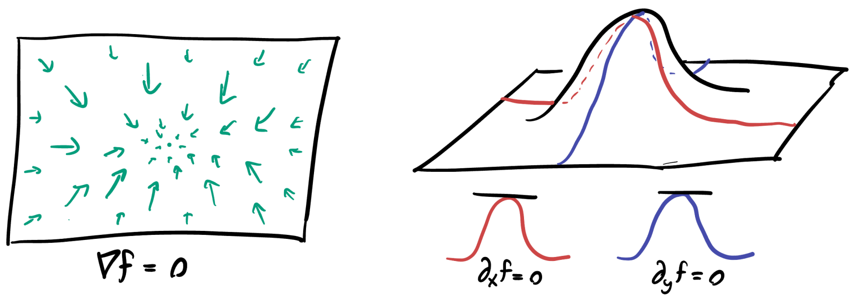


Figure 11.2.: At a local max or min, the gradient is zero.

Theorem 11.1 (The Gradient at Extrema)**.** *At a local max or min, every directional derivative is zero, because the point is a local max or min in* every direction. *In particular, all partial derivatives are zero, so the gradient is zero.*

Definition 11.2 (Critical Points)**.** The critical points of a function are the points where the gradient is zero.

Like in Calculus I, we have to be careful as not all critical points are actually maxima or minima. The standard example there is $y = x^3$ which has $y' = 3x^2$ equal to zero at $x = 0$, even though this is not the location of an extremum but rather a *point of inflection*. Similarly, for multivariable functions the existence of a critical point does not imply the existence of an extremum. The easiest and most common counter-example here is the *saddle*:

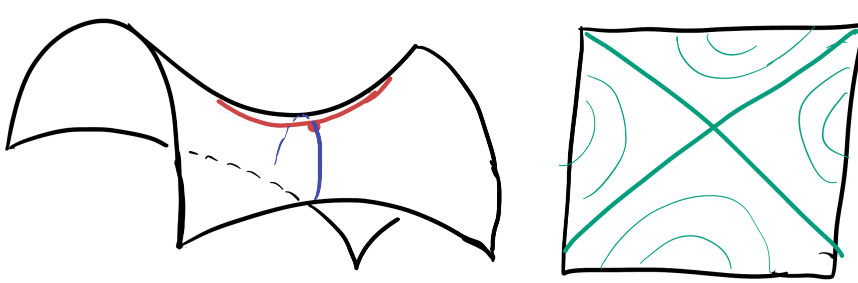


Figure 11.3.: A saddle point also has gradient zero: it's at the intersection of two contour lines - meaning there are two directions where the function has zero derivative. It's also the location of a maximum (in one direction) and a minimum (in the other) meaning the directional derivative is zero here too!

Example 11.1 (Critical Points)**.** Find the critical points of

$$f(x, y) = x^3 + y^3 + 6xy$$

Solving the system of equations arising from setting the gradient to zero is the analog of the first derivative test. What's analog of the second derivative test? In Calculus I, this was looking for the "concavity" of the function, which was simply up or down. But we already know in multiple variables things are more complicated: there are hills, bowls *and* saddles to contend with.

Our tool to see which is the best local description is the *quadratic approximation*, which is particularly simple at a critical point. The zeroth order term is just a constant (which shifts a graph up or down but doesn't affect its shape), and the linear terms are zero - that's the definition of a critical point! Thus *all* we are left with are the quadratic terms, which were determined by the Hessian - the matrix of second derivatives.

Here we recall the *Hessian*, which is a means of organizing all of the second derivative information.

Definition 11.3 (Quadratic Approximation at a Critical Point)**.** Recall the Hessian matrix is the matrix of all second derivatives

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

At a critical point, p, up to a constant the quadratic approximation is built out of the Hessian alone

$$f \approx \text{Const} + \frac{1}{2}\left(f_x x(a,b)(x-a)^2 + 2f_{xy}(x-a)(y-b) + f_{yy}(a,b)(y-b)^2\right)$$

Where here we've used that $f_{xy} = f_{yx}$ to simplify.

To understand if our function has a max, min or saddle at a given critical point, we just need to find a formula in terms of f_{xx}, f_{xy} and $f_{yy}ee$ that determine the shape of this graph. The important tool here is the *determinant* (which we already met when computing cross products)

$$D = \det Hf = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

Where again we've used that $f_{xy} = f_{yx}$ to simplify our formula. The sign of this quantity determines whether the quadratic is a saddle:

Theorem 11.2. *If p is a critical point of $f(x, y)$ and D is the determinant of the second derivative matrix at p, then*

- *p is a saddle if* $D < 0$.
- *p is a minimum if* $D > 0$ *and* $f_{xx} > 0$
- *p is a maximum if* $D > 0$ *and* $f_{xx} < 0$
- *Otherwise, the test is* indeterminant.

It is possible to go *beyond the quadratic approximation* and understand the points this test labels indeterminant, but this requires more complicated mathematics and rarely shows up in real-world applications.

Its helpful to confirm this test is doing the right thing for examples we already understand: so let's take a quick look at a maximum, minimum and saddle:

Figure 11.4.: Local maxes and mins both have $D > 0$.

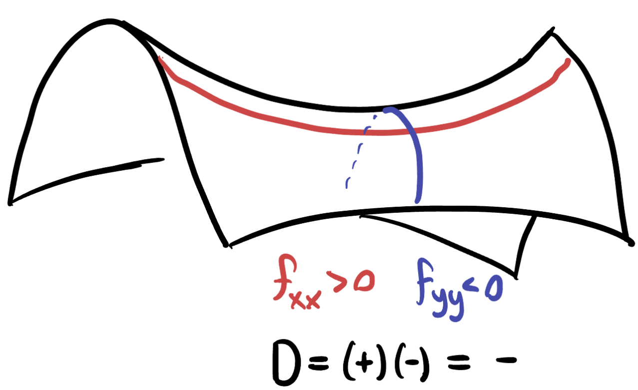


Figure 11.5.: Saddle points have $D < 0$.

11.1. Finding Maxima Minima and Saddles

Example 11.2 ($x^2 + y^2 - 2x - 6y + 14$).

Example 11.3 ($x^3 + y^3 + 6xy$).

Example 11.4 ($2x^3 - yx + 6xy^2$).

11.2. Sketching Multivariate Functions

Having precise mathematical tools to understand the critical points of a function allows us to understand the total behavior of the function - because it gives us the tools to draw a contour plot! Here's an example: say we ran the above computations and found a function with three critical points: a max a min and a saddle.

We plot and label them on an xy plane, and then we can draw little local models of what the contours must look like nearby, since we know the contours for maxes mins and saddles!

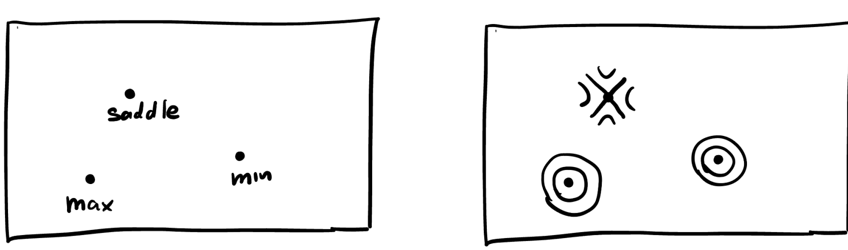


Figure 11.6.: Sketching local information from the critical points.

Next, since there are *no other critical points* we know there isn't anything else interesting going on in our function's behavior. So, we can extend this to a drawing of the contour plot for the whole function by first extending the lines that already exist in a way that they do not cross (if they crossed anything else, that would be representing a *new saddle*- but we know there are none!) And then, we can just fill in contour lines in a non-intersecting way essentially uniquely, so that they create no new saddles or closed loops (which would have a new max or min in their center!)

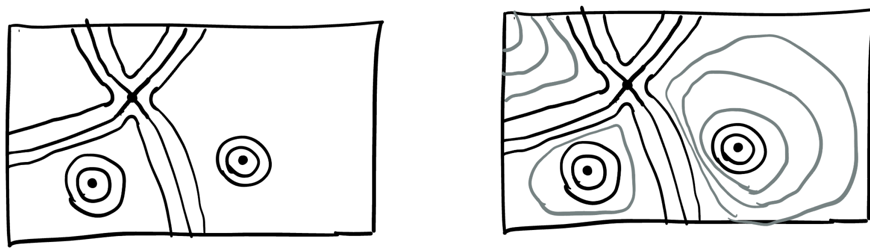


Figure 11.7.: Expanding this to an entire contour plot.

The observation that makes this possible is that nothing strange can happen away from a critical point: if the first derivative is nonzero, then the function is simply increasing or decreasing (in some direction), and the level sets nearby locally look like a set of parallel lines! This is a gateway to a huge amount of modern and advanced mathematics called *morse theory*

https://stevejtrettel.site/code/2023/contour-slicing/

This technique remains important even beyond scalar functions, where changes in contours signify changes in the *topology* of a shape!

https://stevejtrettel.site/code/2023/morse-function/

11.3. Videos

11.3.1. Calculus Blue

https://youtu.be/J1HS6hmBtC0?si=pM8eYtifjLOADcCJ

https://youtu.be/XLe0YMZkh70?si=lMgHpAA-QvMQlJFB

https://youtu.be/7ncgs04-Epw?si=zAChu5xahRhnBm10

https://youtu.be/ZtmglCu_XE8?si=ViVsdognIGyYNcU_

11.3.2. Khan Academy

https://youtu.be/ux7EQ3ip2DU?si=9ssdihUGBJDexoOF

https://youtu.be/8aAU4r_pUUU?si=phRaS8458TaUeHBS

https://youtu.be/nRJM4mY-Pq0?si=bRJpT9axN_QsCiBF

https://youtu.be/m1FhUjMMv30?si=9GQxrx9DVAhd7kpX https://youtu.be/sJo7D74PAak?si=JBmJT4Gelk2G29hb

Example Problem:

https://youtu.be/shWXeUn5BHk?si=4vhAu1mjaXFaf-zO

https://youtu.be/TqslX-bUTD8?si=IH5wv4HwsajoqUnp

11.3.3. Example Problems

https://youtu.be/RqRnKry9L3g?si=uNPUQnwhgfSand_I

https://youtu.be/Yirl8OvO3tU?si=9WtKpDsp7jtDlERl

https://youtu.be/odkaPgWPQGo?si=-8zT4yHo5HR_dryb

https://youtu.be/xl-4T8ak8Eg?si=ptjIrk2jzeQTlX0_

11.3.4. Optimization Example Problems:

https://youtu.be/nb-_bs6eSYo?si=QNnkoVg3FcDY22Oc

https://youtu.be/laKOfcVjrlc?si=cDg2tDsLthqXWZa9

https://youtu.be/PWAWxpG1zjg?si=1BBKqjw_IOubwn2T

12. The Gradient

This set of notes dives deeper into the *geometry* of the first derivative. As a review, we remind ourselves of the notation for the gradient here.

Definition 12.1 (The Gradient)**.** The gradient of a function $f(x, y)$ is

$$\nabla f = \langle f_x, f_y \rangle$$

The gradient of $f(x, y, z)$ is the 3-dimensional vector

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

The symbol ∇ is called *nabla* or *del*, and is a shorthand for the *vector of partial derivative operators*:

$$\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$$

This notation is convenient here as

$$\begin{aligned} \nabla f &= \langle \partial_x, \partial y \rangle f \\ &= \langle \partial_x f, \partial_y f \rangle \\ &= \langle f_x, f_y \rangle \end{aligned}$$

12.1. Directional Derivatives

We have seen that $\partial_x f$ and $\partial_y f$ measure the slope of a multivariate function in the x and y directions, respectively. But what is its rate of change in the direction of an arbitrary unit vector u?

Definition 12.2 (Directional Derivative)**.** The derivative of f in the direction of a unit vector u is denoted $D_u f$ and is defined by the limit

$$\lim_{h \to 0} \frac{f(p + \epsilon u) - f(p)}{\epsilon}$$

Computing this seems difficult. But we can use the Fundamental Strategy of Calculus to save the day! We linearized the surface at a point using the *differential*, which gave the approximation $dz = f_x dx + f_y dy$. Now dx and dy are just placeholders to represent small changes in x, y respectively, so if we are looking for a change in the direction $\langle a, b\rangle$ we have $dx = a$ and $dy = b$, so

$$dz = af_x + bf_y$$

That is, the directional derivative is just a linear combination of the two basic slopes we already know!

Theorem 12.1 (Directional Derivative)**.** *If $u = \langle a, b\rangle$ is a unit vector, then*

$$D_u f(x, y) = af_x(x, y) + bf_y(x, y)$$

All of this carries over to three or higher dimensions: if $u = \langle a, b, c\rangle$ is a unit vector and $f(x, y, z)$ is a three variable function then

$$D_u f = af_x + bf_y + cf_z$$

Any time we have a collection of *sums of products of terms* we should think, 'Is this a dot product?' And in this case it is! If we factor the above equations into a dot product we see the directional derivative is related directly to the gradient.

Theorem 12.2 (Directional Derivatives and the Gradient)**.**

$$D_u f(x, y) = \nabla f \cdot \hat{u}$$

12.2. Geometry of the Gradient

Since we know the interpretation of dot products in terms of angles, we can use the directional derivative formula above to help us understand the direction the gradient points in.

If a vector u makes angle θ with the gradient, we see the directional derivative in direction u is given by

$$D_u f = \nabla f \cdot u = \|\nabla f\|\|u\| \cos\theta = \|\nabla f\| \cos\theta$$

This actually tells us alot!

Theorem 12.3.

- *The gradient points in the direction of maximal directional derivative.*
- *Its magnitude is the directional derivative in that direction*
- *In the orthogonal direction to the gradient, the directional derivative is zero: the function is not changing!*

https://stevejtrettel.site/code/2022/gradient

The last of these facts is so useful on its own, that it gets it's own theorem box:

Theorem 12.4. *The gradient vector is orthogonal to the level sets of a function, and points in the direction of increase.*

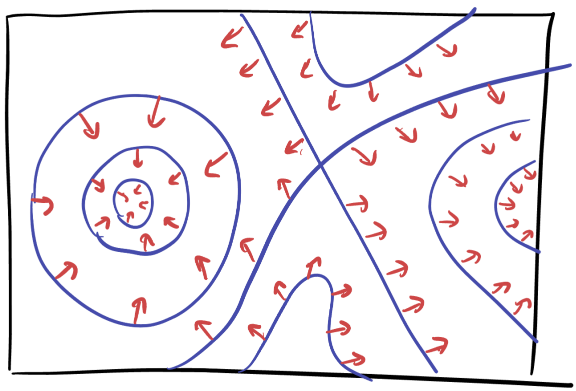


Figure 12.1.: The gradient is orthogonal to level sets.

This is very helpful for understanding a function from its gradient, as it lets us convert between level set understanding and gradient understandings!

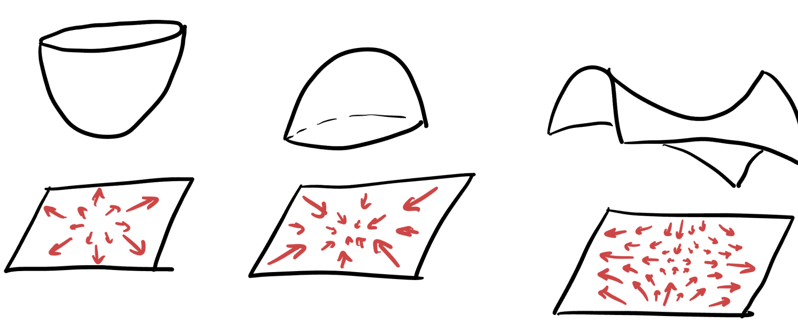


Figure 12.2.: The gradient points in the direction of steepest ascent.

12.2.0.1. The Gradient and Level Sets

When level sets are close to each other, that means the function is steeply increasing or decreasing, so the gradient is *long*. When level sets are far apart, that means the function is only slowly changing, so the gradient is short. Thus, there's an inverse relationship between the length of the gradient and the density of level sets.

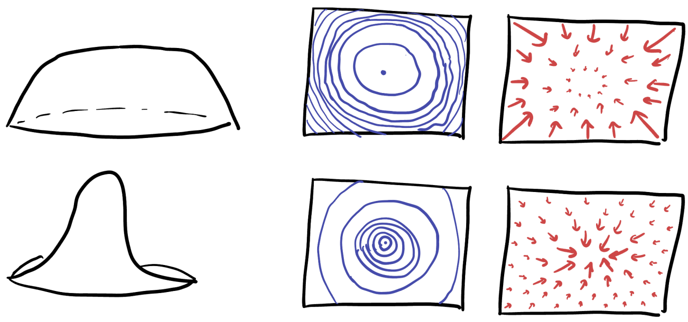


Figure 12.3.: The length of the gradient is inversely proportional to the density of contour lines.

12.2.1. Tangent Planes to Level Sets

Because the gradient is a *normal vector to level sets* we can use the gradient to derive the equation for a tangent plane to a surface! We previously wrote it down of for functions,

$$f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c) = 0$$

But this was just in analogy with the tangent line case. Now, we wish to derive it from our original description of planes, in terms of their *normal vectors*: if p is a point on the plane and n is a normal vector to the plane, the equation

$$n \cdot ((x,y,z) - p) = 0$$

Describes the plane because it says (x,y,z) lies in the plane so long as the vector connecting it to p is orthogonal to n. Now that we know the gradient

$$\nabla f(a,b,c) = \langle f_x(a,b,c), f_y(a,b,c), f_z(a,b,c) \rangle$$

is the normal vector to our plane, we can directly write down the equation for the normal at (a,b,c):

$$\nabla f(a,b,c)\cdot((x,y,z)-(a,b,c))=0$$

and, after computing the dot product we can see it's the same equation we already know!

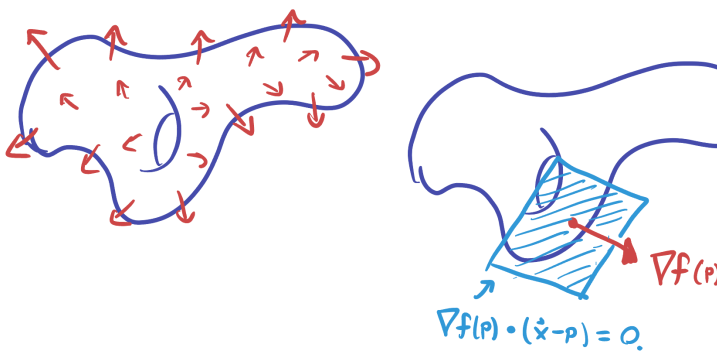


Figure 12.4.: The gradient is normal to level sets, even in 3D. This makes it easy to use the gradient to find the tangent plane to a level set.

But knowing the normal vector also allows us to compute other geometric quantities of interest: such as the **normal line**: the parametric line which intersects a level set orthogonally.

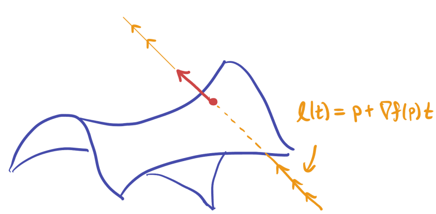


Figure 12.5.: The normal line to a level set in three variables.

This is also immediate: as if we know a point p and a direction vector v, the associated line is $\ell(t)=p+tv$. So here, the point is (a,b,c) and the normal vector is $\nabla f(a,b,c)$ so the normal line is

$$\ell(t)=(a,b,c)+t\nabla f(a,b,c)$$

Example 12.1. Compute the tangent plane and the normal line to $x=y^2+z^2+1$ at $(3,1,-1)$.

First, we re-arrange so that the surface equation is written as a level set: $x-y^2-z^2=1$ with all the variables on one side. Now we can compute the gradient:

$$\nabla f = \langle 1, -2y, -2z\rangle \qquad \nabla f(3,1,-1) = \langle 1,-2,2\rangle$$

This vector and the original point $(3,1,-1)$ immediately determine the plane and line:

$$\langle 1,-2,2\rangle \cdot \langle x-3, y-1, z+1\rangle = 0$$
$$x-2y+2z=-1$$

$$\ell(t) = (3,1,-1) + t\langle 1,-2,2\rangle = (3+t, 1-2t, -1+2t)$$

12.3. Videos

12.3.1. Calculus Blue

https://youtu.be/Xxy_xvbMAew?si=sLgi-Xs2c_aEiEwg

https://youtu.be/tIuGAxwqM5M?si=GLhbfp1WgWi9zHGj

https://youtu.be/tIuGAxwqM5M?si=WK_PHSDEy9YKK4uX

https://youtu.be/SsBiqZ8JtRs?si=v4OuVbr3EM6WZ25M

12.3.2. Khan Academy:

https://youtu.be/tIpKfDc295M?si=2p92m3R1nyE9kXw4

https://youtu.be/_-02ze7tf08?si=A52-NJaNn9HU-jJj

https://youtu.be/N_ZRcLheNv0?si=IKKkK8H-9OthIxcB

Directional derivatives and slope:

https://youtu.be/4tdyIGIEtNU?si=tu21kvEpX00zAiJl

Why the gradient is the direction of steepest ascent:

https://youtu.be/TEB2z7ZlRAw?si=IvKKBQjmEqL3Ej8m

The gradient and Contour Maps:

https://youtu.be/ZTbTYEMvo10?si=8u9asDeBoILbVY97

12.3.3. Example Problems

https://youtu.be/_qAPnUIrLqg?si=d6HSrlMLO3YezfNK

https://youtu.be/i9hhwAZ6hYs?si=5-oO4ScrIfpvLMvU

https://youtu.be/ErZGbQeWlAQ?si=UvQeAp2I12snMMP5

https://youtu.be/xBKhPZ5RgzQ?si=4Asi0yLE8RBPriFb

https://youtu.be/X3UjqMtWq9U?si=i8q8skju38xywdw7

https://youtu.be/GJODOGq7cAY?si=0O7PHxWj-HrH1C2O

13. Constrained Optimization

Realistic optimization problems often involve some sort of a *constraint*:

- What is the best product we can make, with a fixed budget?
- What is the most efficient rocket we can build of a fixed mass?
- What is the least expensive building design, given the external factors of material and labor costs?

Abstractly, all of these questions ask the following: *what are the extreme values of $f(x, y)$ given that we constrain the points (x, y) by some function, $g(x, y) = c$?* In this section, we learn a couple methods to deal with such questions.

13.1. Method I: Reduce Dimension by Substitution

The first method is just to solve the constraint for one of the variables, and substitute it into the function you wish to optimize. This *forces* the inputs to that function to obey the constraint - so now you can just

Example 13.1. Maximize $z = 4 - 2x^2 - 3y^2 + x - y$ subject to the constraint $x + y - 2$.

This also works in higher dimensions: we can take a problem of three variables with one constraint and turn it into a problem of two variables with no constraints:

Example 13.2. Find the maximum value of xyz subject to the constraint $x+y+z = 1$.

Unfortunately, this entire method relies on being able to *solve* the constraint for a given variable, and substitute it in! But this is often impossible. Most relations of two variables can't be solved for one or the other variable indpendently.

13.2. Method II: Lagrange Multipliers

What are we to do, if we cannot substitute the constraint in? It helps to back up and think geometrically here: in a two variable problem, we can draw the function $z = f(x, y)$ as a surface in $\mathbb{R}^3$, and the constraint $g(x, y) = c$ as a curve in the domain (the xy plane). The values of $f(x, y)$ which satisfy the constraint

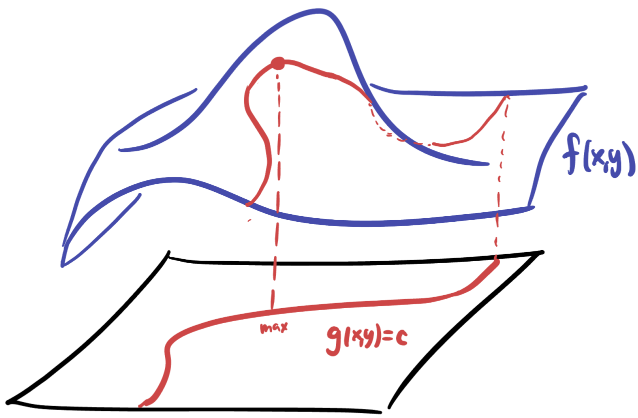


Figure 13.1.: Looking for maxima along a constraint.

It's even more helpful to draw this as a *contour plot* with level sets. The constraint (our hiker's trajectory) still appears as a curve, but we can easily read off exactly where our hiker is going uphill or downhill by looking at *how* they are crossing contours.

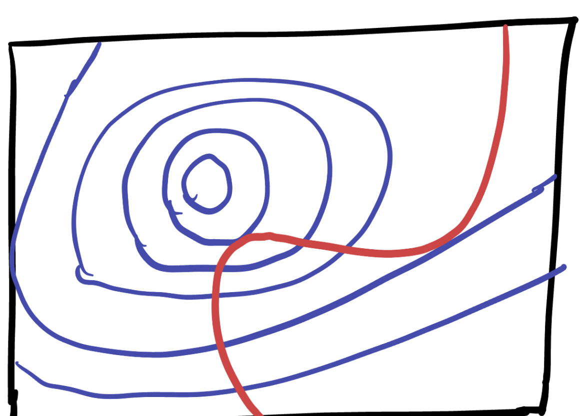


Figure 13.2.: Viewing a constraint problem with level sets.

Whenever the hiker is crossing a contour they are either increasing or decreasing in elevation, and so cannot be at an extremum. Indeed - extrema can only occur at locations where the hiker is *not crossing a level set* - that is, where their path is tangent to a level set! This is the fundamental insight

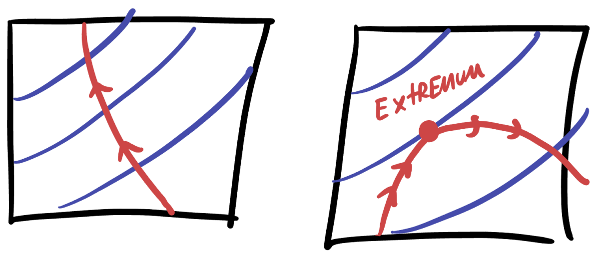


Figure 13.3.: The fundamental insight of constrained optimization.

> Extrema occur when the constraint is tangent to a level set.

Everything follows from this: but the work is in turning this qualitative insight into a system of equations. The first observation we can make is that the constraint itself $g(x, y) = c$ is a level set - just of a different function. So

> Extreme values of f along the constraint $g(x, y) = c$ occur when this constraint level set is tangent to a level set of f.

But this relation of *is tangent to* is still difficult to deal with. To help, we remember that the gradient vector is perpendicular to level sets! Thus, ∇f and ∇g are both perpendicular to their level sets, and thus these vectors must be *parallel* since the level sets are *tangent.*

> Extreme values of f occur along $g(x, y) = c$ whenever ∇f is parallel to ∇g.

Now we've really made some progress! We just need to remember that *parallel* means *are scalar multiples of each other* and give this scalar multiple a name: it's traditional name is the greek letter λ

> Extreme values of f occur along $g(x, y) = c$ when there exists a constant λ with $\nabla f = \lambda \nabla g$.

This is now a fully precise, quantitative claim: it tells us that we can find the constrained extrema by solving a system of equations!

$$\begin{Bmatrix} \nabla f(x, y,) = \lambda \nabla g(x, y) \\ g(x, y) = c \end{Bmatrix}$$

Recalling that the gradient is the vector of partial derivatives, this is just the system of equations

$$\begin{Bmatrix} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = c \end{Bmatrix}$$

This has three variables and three unknowns, so generically we will be able to find some finite number of solutions! Unfortunately there is no general strategy for solving such equations, other than the "substitute one into the other and think about it" technique familiar from precalculus.

It's illustrative to re-do the original example from the substitution section using the method of multipliers. :::{#exm-lagrange-1} Maximize $z = 4 - 2x^2 - 3y^2 + x - y$ subject to the constraint $x + y - 2$. :::

Example 13.3. Maximize $x^2 + 2y^2$ subject to the constraint $x^2 + y^2 = 1$

Example 13.4. Find the maximum value of xyz subject to the constraint $x+y+z = 1$.

13.3. Optimization and Inequalities:

For dealing with an inequality, we need to break the problem into two cases: when the constraint is an equality, we can do the same process we have been learning above, with either Lagrange multipliers or substitution. And, inside of the constraint, we can do our more standard two dimensional optimization (finding critical points, sorting into maxes and mins), and be careful only to consider critical points that are *inside the domain we care about*: if the constraint is x^{2+y}2<1$ and you find a critical point $(3, 0)$ you can ignore it, but the critical point $(1/2, 1/2)$ needs to be considered.

This will result in you having two sets of potential extrema: those occurring on the inside, and those occurring on the boundary. How do you find the absolute max (or min)? That's easy! Just take the largest (or smallest) overall result.

Example 13.5. Maximize $x^2 + 2y^2$ subject to the constraint $x^2 + y^2 \leq 1$

Part IV.

Integration

14. Double Integrals

We now begin a new chapter - after studying in detail various means of studying *change* via multivariate differentiation, we will switch to study *accumulation* via multivariate integration. As Calculus I and II focused on defining the integral of a single variable over a 1-dimensional region (the closed interval $[a, b]$), we will continue in Calculus III to define the integral of multivariate functions over two and three dimensional regions.

14.1. Riemann Sums and Iterated Integrals

In one dimension, an integral measures the area under a graph by breaking in into slices, and adding up approximate areas of each slice, via a Riemann sum, before taking a limit. We will begin with a similar process here, we define the *double integral* of a function $f(x, y)$ over a region R in the plane by a two dimensional Riemann sum.

https://stevejtrettel.site/code/2022/riemann-sum-2d/

This two dimensional Riemann sum works by breaking the region R into small rectangular regions which we will denote ΔA, choosing a point (x_i, y_j) in each such region, and then summing

$$\sum_{i=1}^{N}\sum_{j=1}^{N} f(x_i, y_j)\Delta A$$

As the number of regions goes to infinty, and the size of each rectangle ΔA goes to zero, this becomes an *integral*, with Σ becoming $\int$ and Δ becoming d:

$$\iint_R f(x, y)dA$$

This measures the *volume under the graph* of f above the region R, instead of the area under a curve. But how do we evaluate this thing? We can either add up the volume of each row with constant x first, to get a function of y, and then add these up, or the opposite: first add up in rows of y to get a function of x, then add these up. Either way, we add up all the little volumes, and this gives the total volume under the surface.

You can see this in the animation below: where one of the side bar graphs gives the result of summing along rows first, the other columns, and these two side graphs have the same total area under their curves.

https://stevejtrettel.site/code/2022/fubini

14.2. Rectangular Domains

Let R be the region $a \leq x \leq b$ and $c \leq y \leq d$. Say we want to compute the integral $\iint_R f(x,y)dA$. By the observation above (Fubini's theorem) we can compute this by integrating all the x's first then integrating y, or vice versa:

$$\iint_R f(x,y)dA = \int_a^b \left(\int_c^d f(x,y)dy\right)dx = \int_c^d \left(\int_a^b f(x,y)dx\right)dy$$

https://stevejtrettel.site/code/2023/iterated-integral-cartesian

This is a massive simplification: it means that we can compute two dimensional integrals by just doing two one dimensional integrals, one after the other!

Example 14.1. Evaluate $\iint_R x^2 y\, dA$ for $R = [1,2] \times [3,4]$

Example 14.2. Evaluate $\iint_R x(3-y^2)\, dA$ for $R = [0,2] \times [1,2]$

Oftentimes, the order one performs the integrals in does not matter - both are equally straightforward. But this is not always the case!

Example 14.3. Integrate $y\sin(xy)$ over the region $R = \{0 \leq x \leq 1, 0 \leq y \leq \pi\}$.

Try both orders, see which is easier!

Sometimes, when the function you are integrating is a product of a function of x and a separate function of y, things can simplify even further! If $f(x,y) = g(x)h(y)$ then we may write

$$\iint_R = \int_a^b \int_c^d g(x)h(y)dydx = \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)$$

We get this by realizing that $g(x)$ is a constant with respect to the y integral so we can pull it out: but then once we have done the y integral the result is a constant, so we can pull it out of the x integral!

Example 14.4. Compute

$$\iint_R e^x \sin(y) dA$$

On the region $R = [0, \pi/2] \times [0, \pi/2]$.

This is essentially all there is to the theory of multiple integrals when the domain is a box (where all variables are bounded by constants). Indeed, we will shortly meet triple integrals and see that everything remains precisely the same!

$$\int_R g(x,y,z) dV = \int_a^b \left(\int_c^d \left(\int_e^f g(x,y,z) dz \right) dy \right) dx$$

However, before going there we will continue on and look at more general double integrals: what happens when the region R is not a box?

14.3. Variable Boundaries

In one variable calculus, the only sort of region over which you could perform an integral is a single interval. But in two variables, the regions of the plane over which you could wish to integrate are much more varied!

We have learned how to deal with rectangular regions by *slicing* - and this same technique will serve us well in many other cases. To start, we won't focus on completely general regions, but rather on regions where the top and bottom are bounded by functions of x:

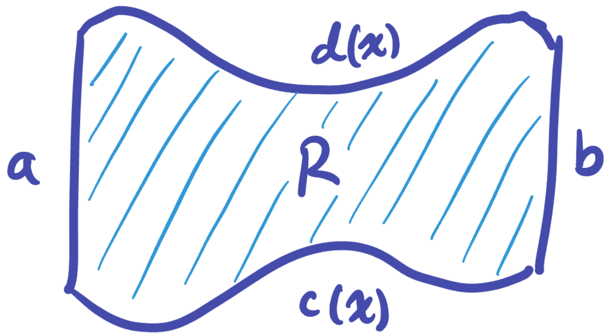


Figure 14.1.: A region where the top and bottom boundaries are functions of x

Here, if we slice with respect to y first, our vertical slices will each be of a different length - but they will still always be intervals (the top and bottom are *functions* meaning they pass the vertical line test - so each intersects the vertical strip exactly once).

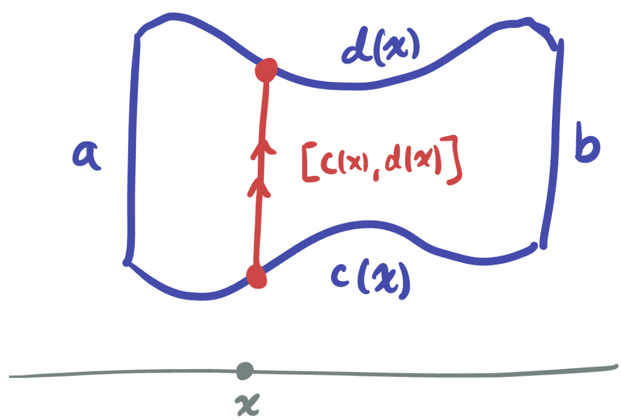


Figure 14.2.: Vertical slices of such a region are intervals.

At the fixed value x, what is the interval we are integrating over? Well, it runs from the bottom function $a(x)$ to the top function $b(x)$, and so the integral along this slice is

$$\int_{c(x)}^{d(x)} f(x,y)dy$$

Then all that remains is to integrate this along the y direction:

$$\iint_R f(x,y)dA = \int_a^b \int_{c(x)}^{d(x)} f(x,y)dydx$$

https://stevejtrettel.site/code/2023/iterated-integral-variable-bounds

Exercise 14.1. Find the integral of $x + 2y$ over the region R

$$R = \{(x,y) \mid 0 < x < 2 \quad x^2 - 2 < y < x\}$$

Sometimes the x bounds don't even need to be given explicitly-they are just the region between where the curves intersect:

Exercise 14.2. Find the volume above the xy plane under the graph of $z = x^2 + y^2$, within the region R bounded by $y = 2x$ and $y = x^2$.

There's nothing special about slicing with respect to the x direction, we can also do integrals by slicing with respect to fixed y, and integrating dx first. Indeed, the above example can be redone this way no problem!

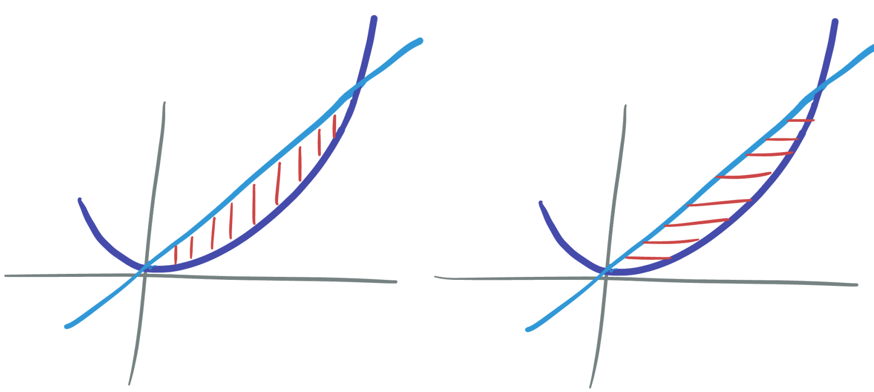


Figure 14.3.: Slicing the same region either vertically or horizontally.

Exercise 14.3. Find the volume above the xy plane under the graph of $z = x^2 + y^2$, within the region R bounded by $y = 2x$ and $y = x^2$, this time first slicing horizontally, with constant y.

However, not every example is just as easy both ways. For example the following integral is easy to write down sliced with respect to y, but harder when sliced first with constant x:

Exercise 14.4. Integrate $x + y$ on the region R determined by

$$R = \{(x,y) \mid -2 < y < 2 \quad y^2 - 1 < x < 3\}$$

14.3.1. Changing the Order of Integration

To do the last integral instead with respect to slices of constant x (so, slices in the y direction) we would need to solve for the y bounds as a function of x,

Exercise 14.5. Set up the integral of $x + y$ on the region

$$R = \{(x, y) \mid -2 < y < 2 \quad y^2 - 1 < x < 3\}$$

Where slicing is first done at constant xs.

One of the most important things about setting up a double integral correctly is thinking through which order of integration will be more useful, and why. Sometimes, one way of slicing will lead to an impossible integral, but the other way will be easy!

Example 14.5. Compute the integral

$$\int_0^1 \int_x^1 \sin(y^2) dy dx$$

by switching the order of integration first.

14.4. Combining Integrals

Just like there is a *subdivision rule* for one dimensional integrals,

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$$

There is a similar rule for double integrals: if you break the domain into two regions, the double integral over the whole thing is the sum of the double integrals over each. In symbols: if $R = R_1 \cup R_2$, then

$$\iint_R f dA = \iint_{R_1} f dA + \iint_{R_2} f dA$$

This lets us perform integrals we otherwise could not, by breaking the domain down into simpler pieces, which we can then slice with respect to either x or y.

14.5. Video Resources

14.5.1. Introduction to Iterated Integrals

https://youtu.be/qv3GzyemweM?si=B0Ct0Rx5_Vp7APB7

https://youtu.be/JXh9AQkKmsw?si=upuVz8o_ZHccNwqa

https://youtu.be/gifQWtTWqEY

14.5.2. Examples Evaluating Double Integrals

https://youtu.be/-0l7PA_A1mE?si=I-ro7r1qizEZuYcH

https://youtu.be/fvK0xHVzYbw?si=Eeocrsxz8y1oeprv

https://youtu.be/pI6cjKQ7Bec

https://youtu.be/vcYBKIo9bAI?si=Nfqic9SmEE9Kme0D

14.5.3. Changing the Order of Integration

https://youtu.be/cVMqWW0p6MM?si=Pvm7gQ0pwWnXcI8

https://youtu.be/LUvynduoUX0

15. Triple Integrals

Triple integrals follow a very similar general theory to double integrals: starting with a function $f(x, y, z)$ on $\mathbb{R}^3$, we define the integral over a region E by breaking that region into small cubical volumes of size dV and building a 3-dimensional riemann sum. Taking the limit gives the triple integral, or

$$\iiint_E f\, dV$$

To evaluate such an expression, we need to break the integral into slices, and evaluate them one at a time. Such slicing relies on understanding the volume element in three dimensions, which is the volume of an infinitesimal box. Since a box's volume is given by length width and height, we can express dV as a product of three infinitesimal lengths:

$$dV = dxdydz$$

This lets us separate the triple integral into three consecutive integrals: first dx, then dy then dz. Or, because the order of multiplication doesn't matter, we could do the integral in any of the other six possible orders

$$dxdydz = dxdzdy = dydxdz$$

$$= dydzdx = dzdxdy = dzdydx$$

15.1. Different Bounds:

Because we are going to evaluate a triple integral as three iterated integrals, we can *reuse* a lot of what we learned about double integrals along the process. In particular, we can think of the process of computing a triple integral as first choosing one of the directions to integrate, and then treating the two remaining directions as a double integral over a region parameterizing all the slices (if we integrate z, each slice has a different x, y coordinate, so the region parameterizing the slices is a side of our 3D region in the xy direction).

$$\iiint_E f dV = \iint_R (\text{Slice})\, dA = \iint_R \left(\int_{z_{\text{start}}}^{z_{\text{stop}}} f\, dz \right) dA$$

This allows us to think about triple integrals as not a new thing, but just *adding one more direction* to a process we already understand well.

15.1.1. Boxes

When the domain $E \subset \mathbb{R}^3$ is a coordinate box, described as

$$E = \{(x, y, z) \mid a \leq x \leq b\, c \leq y \leq d\, e \leq z \leq f\}$$

This triple integral splits into an iterated integral with constant bounds:

$$\iiint_E F dV = \int_a^b \int_c^d \int_e^f F dz dy dx$$

This integral could be done in any of the six possible orders, as all the bounds are constants, no order will be easier or harder than any other.

Example 15.1.

$$\iiint_E xe^y\, dV \qquad E = \{1 \leq x \leq 2, 0 \leq y \leq 1, 2 \leq z \leq 5\}$$

First we choose an order: say we integrate dz first. These bounds go from 2 to 5 so the integral along our slice is

$$\text{Slice} = \int_2^5 xe^y dz = xe^y z\Big|_2^5 = xe^y(5-2) = 3xe^y$$

Now we just have to do the double integral of this over the rectangle $R = \{1 \leq x \leq 2, 0 \leq y\}$ containing all of the slices:

$$\iiint_E xe^y dV = \iint_R 3xe^y dA$$

Again we choose a direction to slice: starting with dy, we decompose this into an integrated integral as

$$\iint_R exe^y dA = \int_1^2 \int_0^1 3xe^y\, dy dx$$

$$= \int_1^2 3x \left(\int_0^1 e^y dy \right) dx = \int_1^2 3x \left(e^y\Big|_0^1 \right) dx$$

$$= \int_1^2 3x(e-1) dx = \frac{3}{2} x^2 (e-1)\Big|_1^2 = \frac{3}{2}(4-1)(e-1)$$

15.1.2. Variables in One Bound

If the domain E is described so that its bounds in at two of the variables are constants, and the third set of bounds are variables, then there is a preferred order in which to integrate. In particular, we know the final answer must be a number so we cannot have variables in the outermost set of bounds, and must be done earlier: the easiest situation is just to do it first!

Example 15.2. For example, consider the following domain E:

$$E = \{(x, y, z) \mid 0 \leq x \leq 2, 0 \leq y \leq 3, 0 \leq z \leq x + y\}$$

Here the z bound is different depending on which point (x, y) you are at, so we do the z-integral first.

$$\text{Slice} = \int_0^{x+y} f\, dz$$

Since both the x and y bounds are constants, the remaining region in the xy plane is a *rectangle*, and we know how to integrate over these.

$$\iiint_E f\, dV = \iint_R \text{Slice} dA = \int_0^2 \int_0^3 \text{Slice} dy dx$$

Putting this all together gives representation of the triple integral as an *iterated integral*:

$$\int_0^2 \int_0^3 \int_0^{x+y} f\, dz dy dx$$

We then evaluate this triple integral as three one dimensional integrals from Calculus I.

15.1.3. Variables in Two Bounds

For more complicated domains, its possible that variables will appear in two of the bounds. (Because the final answer must be a number, we know the outer bounds must be constants, so they cannot appear in all three bounds).

In such cases, the innermost integral can have bounds depending on two variables (the next two to be integrated), and the middle integral can have bounds depending on the outermost integral. This way, at each stage the function only has variables left in it that are still going to be integrated away, and the result is a number. In this case, there is only *one* possible order in which the integral can be performed!

Example 15.3. Here's an example: if E is the following region

$$E = \{-y \le x \le yz, 0 \le y \le z+1, -1 \le z \le 1\}$$

Looking at the bounds, we notice the following:

- The x bounds depend on both y and z
- The y bounds depend on z
- The z bounds are constants.

This suggests a most natural order of integration: we start with the x integral as our first slice, as its bounds depend on the remaining integrals y and z. Then we do the y integral, as its bounds depend on the remaining integral z. Finally we do the z integral as its bounds are constant!

This gives us a direct way to write our integral as an iterated integral, which we can then solve by doing three integrals from Calculus I and II.

$$\iiint_E f\,dV = \int_{-1}^{1}\int_{0}^{z+1}\int_{-y}^{yz} f\,dxdydz$$

Sometimes we have to do some work to *sovle* for the bounds given equations describing the region E. Below is one such example:

Example 15.4. Write the integral of f over the region E as an iterated integral, where E is bounded by the xy plane and the surface $z = 1 - x^2 - y^2$.

Since the xy plane is given by $z = 0$, the surfaces we are given directly describe the z bounds, which suggests we begin with the dz integral.

$$\text{Slice} = \int_0^{1-x^2-y^2} f dz$$

But now we are left to *discover the remaining bounds for ourselves!* How do we do this? The region R over which we need to add up the z slices can be described as the points (x, y) where the z bounds define some interval of integration. The boundary of this region is where the z bounds collapse to become equal! That is, R is cut out by what we get from equating the bounds

$$z = 1 - x^2 - y^2 \qquad \text{and} z = 0$$

This implies $x^2 + y^2 = 1$, or the region R is bounded by the unit circle. We are now to a *double integral* where we must again choose an order of slicing. If the bounds we had were given in $x =$ or $y =$ form such a choice might be obvious, but here there's no natural one so we just need to choose, and solve for the correct bounds.

Choosing to do y next, we solve to get $y = \pm\sqrt{1-x^2}$ as the upper and lower bounds of the integral. The interval for x integration is similarly given by the points on the line where these y bounds define an interval: so its boundary is where they collapse to be equal! Setting $\sqrt{1-x^2} = -\sqrt{1-x^2}$ we see the only solution is where this quantity is zero, so $\sqrt{1-x^2} = 0$ or $x^2 = 1$ so $x = \pm 1$. This gives the interated integral

$$\iiint_E f dV = \iint_R \text{Slices} dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \text{Slices} dydx$$

Finally putting it all together gives us a description as an iterated integral, which we could solve by doing three single variable integrals:

$$\iiint_E f dV = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{1-x^2-y^2} f dzdydx$$

This last example looks rather intimidating with the square roots in the bounds, as this means the final integral we have to do will likely involve $\sqrt{1-x^2}$ after we plug these in. Such integrals often require trigonometric substitution to evaluate, and while this is often possible, we will learn in the next section a way to simplify such integrals with a *change of coordinates*.

15.2. Describing the Bounds:

The thing that makes triple integrals challenging is not doing the integrals (its just three 1D integrals) or even choosing the order to do them in (as we saw above, once you have described the domain in terms of x, y, z, its easy to decide which order to do the integral.) The difficult part is often just describing the bounds themselves!

This is mostly because visualizing 3D geometry takes some training to get used to! It's helpful to look through many examples: please remember to be using the book (**chapter 15**), where each chapter is essentially just a giant list of example problems fully worked out! Additionally, Here is another collection of fully worked examples online:.

This video does a good job of explaining the process (and advocating you going and doing lots of your own practice!)

https://youtu.be/LFw0mSbUC8c?si=0HpToKsI7QuJA1s-

15.3. Video Resources

https://youtu.be/zOT1nnUGxHI?si=ruORgI7-uwtFHFRa

https://youtu.be/zER-E__bTpM?si=FRO4Z6_lHcCWmUuM

https://youtu.be/ZIn1rgZVPFw?si=08pkpPqOOaLv2cQ5

https://youtu.be/7iy83x8bv6o?si=m__7ic-UGmrjFFIR

16. Integrals & Coordinates

We've seen previously that certain double and triple integrals are particularly challenging because their bounds contain complicated expressions like $\sqrt{1-x^2}$, which lead to you having to do an *integral* of functions containing things like $\sqrt{1-x^2}$ which leads to difficult trigonometric substitutions, or worse.

These sort of expressions come up when integrating over circular, cylindrical and spherical regions, because these are all described with equations like $x^2+y^2=1$ or $x^2+y^2+z^2=1$ in $\mathbb{R}^2$ or $\mathbb{R}^3$. And this chapter is the bearer of good news: the reason these integrals look hard at first is that the cartesian coordinates x, y, z are not a good way to work with them. But, after adapting our viewpoint, all the square roots melt away and these integrals become straightforward to compute!

The reason is that cartesian coordinates are good for describing flat objects: the surfaces where one variable is held constant describe lines or planes. Thus, integrals over rectangles and boxes are easy in cartesian coordinates: their bounds are constants! To make integrals over circles, cylinders and spheres easy, we need to find coordinates for which circles, cylinders and spheres are described by constants. If we can change our perspective to work with these coordinates, we will be able to turn an integral with difficult bounds into a different integral with constant bounds - but the same overall value.

16.1. Polar Coordinates

Polar coordinates are a means of representing the plane using distance r from the origin, and angle θ from the x-axis.

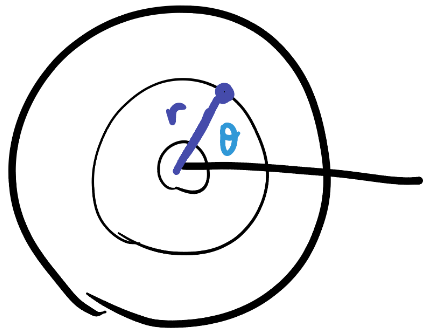


Figure 16.1.: Polar Coordinates definition.

Using trigonometry, we can relate these to the usual x and y coordinates we are used to.

Definition 16.1 (Polar Coordinates)**.** Polar coordinates on the plane are the coordinates r, θ where r measures the distance from the origin, and θ measures the angle from the $x-$axis. The conversion from cartesian to polar coordinates is given by the functions

$$x = r\cos\theta \qquad y = r\sin\theta$$

Definition 16.2 (dA in Polar Coordinates)**.** The area element dA was expressed in cartesian coordinates as $dA = dxdy$ by drawing a small rectangle and taking length times width. The same approach succeeds in polar coordinates, where we draw a small rectangle using an infinitesimal angle $d\theta$ and an infitesimal change in radius dr

PICTURE

Here we must be careful however, as while dr does represent one side of the rectangle, $d\theta$ is an *angle* not a side length. The corresponding side length is an arc of a circle of radius r, and since arc length is proportional to radius we see $ds = rd\theta$. Together this gives

$$\begin{aligned} dA &= (\text{length})(\text{width}) = (ds)(dr) = (rd\theta)(dr) \\ &= rdrd\theta \end{aligned}$$

This lets us do a double integral in polar coordinates by first doing an r integral, and then a θ integral or vice-versa:

$$\iint_R f\,dA = \int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f\,rdrd\theta$$

Just like in cartesian coordinates, you can view this as slicing in teh r and θ directions, and integrating the results.

https://stevejtrettel.site/code/2023/iterated-integral-polar

Starting from an integral with cartesian coordinates x, y, there is a straightforward procedure to convert to polar:

- Convert the function to polar coordinates: substitute $x = r\cos\theta$ and $y = r\sin\theta$ and simplify (remember $x^2 + y^2 = r^2$).
- Substitute dA or $dxdy$ for the polar area unit $dA = rdrd\theta$.
- Rewrite the bounds of integration in terms of polar coordinates.

Now you just have a standard iterated integral (but with variables named r and θ instead of x and$ y$.) This can be computed as normal: just doing one integral at a time.

Example 16.1. Let D be the region inside the unit circle in the plane. Compute the integral

$$\iint_D x^2 + y^2 dA$$

With the bounds being the unit circle, if we slice the integral using cartesian coordinates we will get

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} x^2 + y^2\, dydx$$

This turns out to be a challenging integral to do, so instead we decide to try *polar coordinates*. Converting the bounds first shows this is a good idea: for the unit circle we have

$$0 \le r \le 1 \qquad 0 \le \theta \le 2\pi$$

The bounds are *constant*! Converting the function

$$x^2 + y^2 = (r\cos\theta)^2 + (r\sin\theta)^2 = r^2$$

So our integral becomes

$$\iint_D x^2 + y^2 dA = \int_0^{2\pi}\int_0^1 r^2\, rdrd\theta$$

This is easily evaluated:

$$\int_0^1 r^3 dr = \left.\frac{r^4}{4}\right|_0^1 = \frac{1}{4}$$

$$\int_0^{2\pi} \frac{1}{4} d\theta = \frac{2\pi}{4} = \frac{\pi}{2}$$

In this case polar coordinates led to an integral with only r, and no θ dependence. In general there will be sines and cosines in the resulting integral, which will require us to remember techniques for trigonometric integrals from Calculus II.

:::{#thm-trigonometric integrals} If an integral contains an even power of sin or cos, use the half angle identities (perhaps repeatedly) to decrease the power

$$\cos^2\theta = \frac{1+\cos 2\theta}{2} \qquad \sin^2\theta = \frac{1-\cos 2\theta}{2}$$

For example:

$$\int \sin^2\theta d\theta = \int \frac{1}{2}(1-\cos 2\theta)d\theta = \frac{1}{2}\left(\theta - \frac{1}{2}\sin 2\theta\right) + C$$

If an integral contains an odd power of sine or cosine, save one of the factors and convert the others from sine to cosine (or vice versa) using the pythagorean identity $\sin^2 + \cos^2 = 1$. This sets it up for a u-substitution. For example:

$$\int \cos^5\theta d\theta = \int (1-\sin^2\theta)^2 \cos\theta d\theta$$

$$= \int (1-u^2)^2 du = \int 1 - 2u^2 + u^4 du = \sin\theta - \frac{2}{3}\sin^3\theta + \frac{1}{5}\sin^5\theta$$

:::

Application: Integrating the Gaussian

https://youtu.be/j9KRcCB8pxU?si=3QZq6l8bXSiRq_F1

16.2. Cylindrical Coordinates

Cylindrical coordinates are just the natural three dimensional extension of polar coordinates, where we use r, θ, and z.

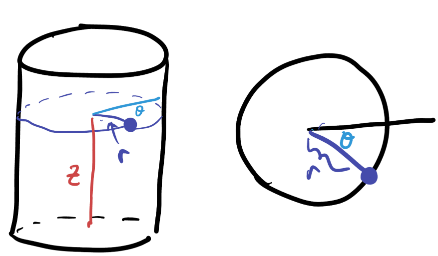


Figure 16.2.: Cylindrical coordinates definition

Definition 16.3 (Cylindrical Coordinates)**.** Measure two directions in space using polar coordinates, and the orthogonal direction with its standard Cartesian axis. If we convert the xy plane to polar, this means

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

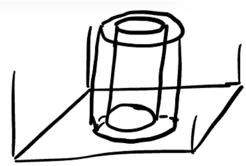
The volume element here is just the polar area element times dz:

Definition 16.4 (Volume in Cylindrical Coordinates)**.**

$$dV = (dA)dz = rdrd\theta dz$$

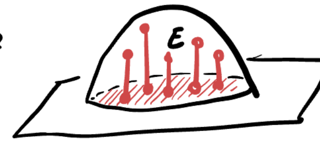
Examples are easier to understand if you *draw the regions* along the way, so I've done some on my iPad to post below:

Cylindrical Coordinates



Convert 2 of the coordinates to polar.

$$\iiint_E z\,dV \qquad \underbrace{z=0 \qquad z = 1 - x^2 - y^2}_{E}$$



$$\iint \int_0^{1-x^2-y^2} z\,dz\,dA$$

One first ↗

$$z^2/2 \to \frac{(1-x^2-y^2)^2}{2}$$

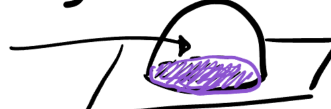
$$\iint_R \frac{(1-x^2-y^2)}{2}\,dA$$

We should Do polar! ✓

$$\iiint_E z\,dV \qquad \text{Bounds } \left\{ z=0 \quad z = 1-x^2-y^2 \right\}$$

Do z first. ↗

$$\begin{bmatrix} 0 \le z \le 1-r^2 \\ 0 \le r \le 1 \\ 0 \le \theta \le 2\pi \end{bmatrix} \qquad \begin{bmatrix} dV = dx\,dy\,dz \\ = (r\,dr\,d\theta)\,dz \end{bmatrix}$$



Circle in xy so convert xy to polar, Leave z alone!

$$\int_0^{2\pi}\int_0^1\int_0^{1-r^2} z\,dz\; r\,dr\,d\theta = \quad \frac{z^2}{2}\Big|_0^{1-r^2} = \frac{(1-r^2)^2}{2}$$

$$\int_0^{2\pi}\int_0^1 \frac{(1-r^2)^2}{2}\,r\,dr\,d\theta$$

$$\frac{1-2r^2+r^4}{2}\cdot r = \frac{1}{2}\left(r - 2r^3 + r^5\right)$$

$$\int_0^1 \frac{1}{2}(r-2r^3+r^5)\,dr = \frac{1}{2}\left(\frac{r^2}{2} - \frac{r^4}{2} + \frac{r^6}{6}\right)\Big|_0^1 = \frac{1}{2}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right) = \frac{1}{12}$$

$$\int_0^{2\pi} \frac{1}{12}\,d\theta = \frac{2\pi}{12} = \boxed{\frac{\pi}{6}}$$

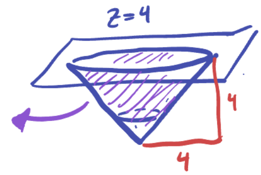
☺

A second example:

Cylindrical Coordinates: Example II

$$\iiint_E x^2 \, dV \qquad E = \{ z \geq \sqrt{x^2+y^2} \, , \, z \leq 4 \}$$

$z \geq r$



*Idea: Convert x & y to polar!!

① Convert function:

$x = r\cos\theta$, $y = r\sin\theta$, leave z alone.

$x^2 = (r\cos\theta)^2 = r^2\cos^2\theta$

② Convert dV

$dV = dx\,dy\,dz$ (to polar; dz: leave alone!) $= r\,dr\,d\theta\,dz$ "dV in cylindrical coordinates"

③ z goes first: $r \leq z \leq 4$, $0 \leq r \leq 4$, $0 \leq \theta \leq 2\pi$

$$\int_0^{2\pi}\int_0^4\int_r^4 r^2\cos^2\theta \, dz \, r\,dr\,d\theta$$

$$r^2\cos^2\theta \, z \Big|_r^4 = r^2\cos^2\theta(4-r)$$

$$\int_0^{2\pi}\int_0^4 r^2\cos^2\theta(4-r) \, r\,dr\,d\theta$$

← don't forget me!!

$r^3(4-r) = 4r^3 - r^4$

$$\int_0^{2\pi}\int_0^4 \cos^2\theta\,(4r^3 - r^4)\,dr\,d\theta$$

$$\cos^2\theta\left(r^4 - \frac{r^5}{5}\right)\Big|_0^4 = \cos^2\theta\left(4^4 - \frac{4^5}{5}\right)$$

$256/5$

$$\int_0^{2\pi} \frac{256}{5}\cos^2\theta\,d\theta = \frac{256}{5}\int_0^{2\pi}\cos^2\theta\,d\theta$$

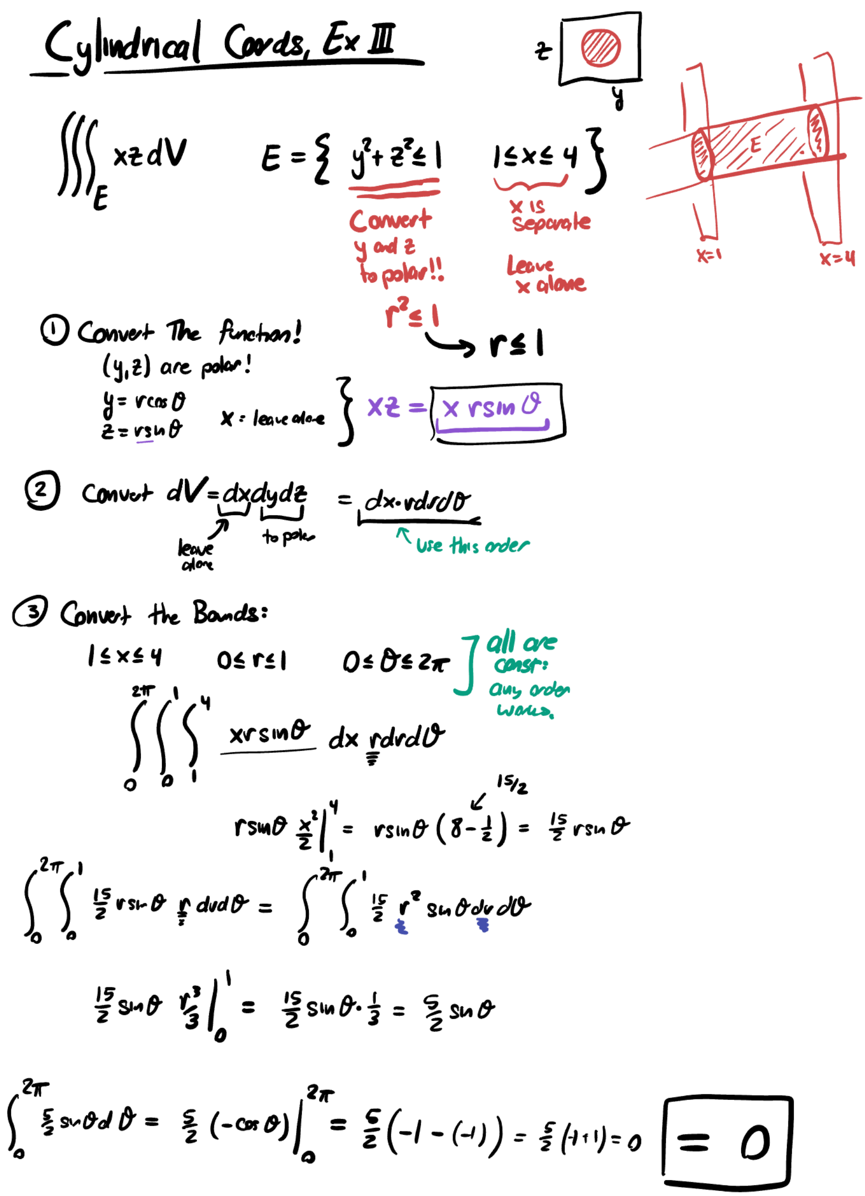
$\cos^2\theta = \frac{1+\cos 2\theta}{2}$

$$\frac{256}{5}\int_0^{2\pi} \frac{1}{2} + \frac{1}{2}\cos 2\theta \, d\theta$$

$$\left(\frac{1}{2}\theta + \frac{1}{2}\cdot\frac{1}{2}\sin(2\theta)\right)\Big|_0^{2\pi} = \boxed{\frac{256}{5}\pi}$$

$\pi + 0 - 0$

And, a third example:



16.3. Spherical Coordinates

Spherical coordinates is a coordinate system in $\mathbb{R}^3$ where we represent a point with latitude, longitude, and radius.

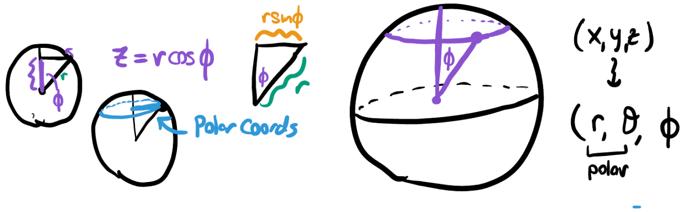


Figure 16.3.: Defining spherical coordinates.

Definition 16.5 (Spherical Coordinates)**.**

$$x = r\cos\theta\sin\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\sin\phi$$

Using these coordinate definitions we can compute the volume element in spherical coordinates: it'll be a product of the length in the r direction, the length in the θ direction and the length in the ϕ direction.

- Length in the r direction is dr.
- Circles in the θ direction (lines of longitude) have circumference $2\pi r\sin\phi$. Thus a small amount of angle has length $r\sin\phi d\theta$.
- Circles in the ϕ direction are all longitudes on the sphere, of length $2\pi r$. Thus a small bit of angle has length $rd\phi$.

Definition 16.6 (Volume in Spherical Coordinates)**.**

$$\begin{aligned} dV &= dr(r\sin\phi d\theta)(rd\phi) \\ &= r^2\sin\phi drd\theta d\phi \end{aligned}$$

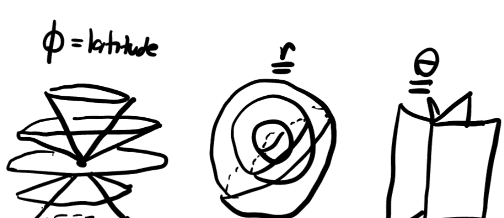
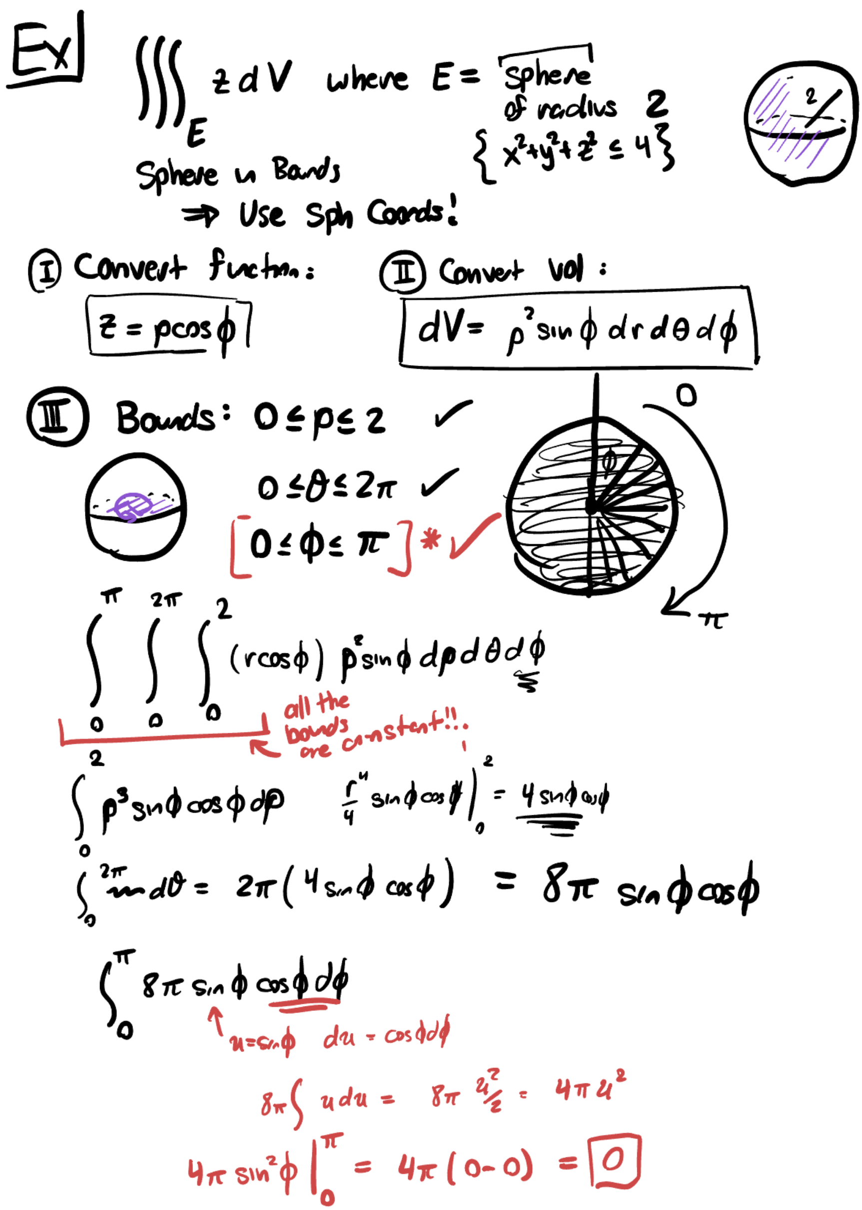


Figure 16.4.: Surfaces of r, θ, ϕ = constant.

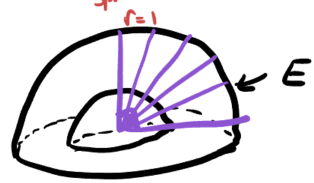
Again, examples are easier when you can draw out the bounds so I've done some handwritten ones below:

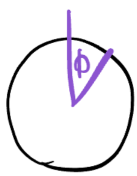


And a second

Find the volume of $E = \{ x^2+y^2+z^2 \le 4 \quad x^2+y^2+z^2 \ge 1 \quad z \ge 0 \}$

sphere $r=2$ sphere $r=1$





In spherical coords:

Bounds
$$\begin{bmatrix} 1 \le \rho \le 2 \\ 0 \le \theta \le 2\pi \\ 0 \le \phi \le \frac{\pi}{2} \end{bmatrix}$$

$$\text{Vol} = \iiint_E 1\,dV$$

$$dV = \rho^2 \sin\phi \, dr\, d\theta\, d\phi$$

$$\text{Vol} = \int_0^{\frac{\pi}{2}} \int_0^{2\pi} \int_1^{2} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$$\rho^2 \to \frac{\rho^3}{3}\Big|_1^2 = \left(\frac{8}{3} - \frac{1}{3}\right) = \frac{7}{3}$$

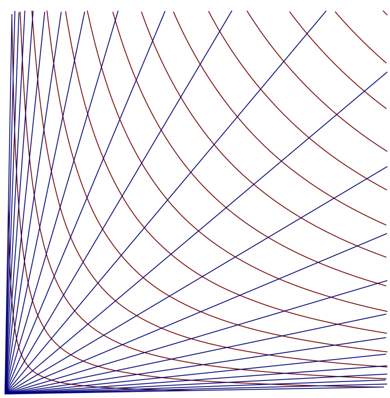
$$\int_0^{2\pi} \frac{7}{3}\sin\phi \, d\theta = 2\pi \cdot \frac{7}{3}\sin\phi = \frac{14\pi}{3}\sin\phi$$

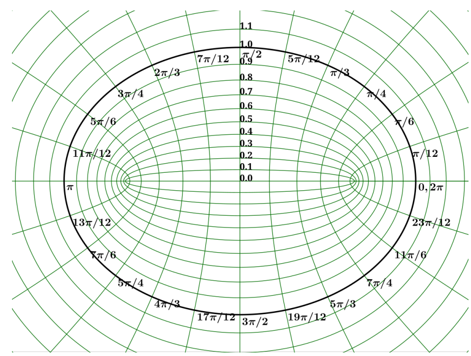
$$\int_0^{\pi/2} \frac{14\pi}{3}\sin\phi \, d\phi = -\frac{14\pi}{3}\cos\phi \Big|_0^{\frac{\pi}{2}}$$

$$= \boxed{\frac{14\pi}{3}}$$

16.4. General Coordinate Transformations

We have spent some time understanding polar and spherical changes of variables, but these are merely the beginning of a rich collection of coordinate changes. There are *elliptical coordinates*, which are helpful when a problem involves ellipses, *hyperbolic coordinates* for problems involving hyperbolas, and many more





Because coordinates help simplify a problem by *making the bounds easier* many of you will see lots of different coordinate systems in your future, especially in physics and engineering.
In this one semester course we will not have the time nor need to deep dive into any other specific examples, but we will take a brief look at the general theory, and see how one can transform *any* integral into *any* coordinates.

When we computed double integrals in polar coordinates, we used the change of variables

$$x = r\cos\theta, \quad y = r\sin\theta$$

to rewrite a region in the xy-plane in terms of new coordinates r and θ. In many examples, this change of variables dramatically simplified both the region of integration and the integrand. The key idea is that if we understand the geometry of a region better in some new coordinate system, we can rewrite the integral accordingly, as long as we account for how area is distorted by the change of coordinates.

This motivates the more general question: given a change of variables

$$x = g(u, v), \quad y = h(u, v),$$

can we express a double integral over a region R in the xy-plane as an integral over a region S in the uv-plane? To answer this, we need to understand how small area elements transform under the map $(u, v) \mapsto (x, y)$.

A Motivating Example

Suppose we are integrating over the region R in the xy-plane bounded by the lines $y = x$, $y = 2x$, $xy = 1$, and $xy = 2$. This region is awkward to describe in Cartesian coordinates, but if we define new variables

$$u = xy, \quad v = \frac{y}{x},$$

then the boundaries become $u = 1$, $u = 2$, $v = 1$, and $v = 2$ — a rectangle in the uv-plane!

This illustrates the power of a good change of variables: a complicated region in xy becomes a simple rectangle in uv. We can't yet evaluate an integral in these coordinates, though, because we need to understand how area is affected by the change of variables.

Deriving the Jacobian in Two Dimensions

Let $(x, y) = (g(u, v), h(u, v))$ be a smooth change of variables. To compute how area changes, consider a small rectangle in the uv-plane with corners at (u, v), $(u + \Delta u, v)$, $(u, v + \Delta v)$, and $(u + \Delta u, v + \Delta v)$. The image of this rectangle under the transformation is approximately a parallelogram in the xy-plane spanned by the vectors

$$\frac{\partial(x, y)}{\partial u}\Delta u \quad \text{and} \quad \frac{\partial(x, y)}{\partial v}\Delta v.$$

The area of this parallelogram is given by the magnitude of the determinant

$$\left|\frac{\partial(x, y)}{\partial(u, v)}\right| = \left\|\begin{matrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{matrix}\right\|.$$

This determinant is called the *Jacobian* of the transformation. It measures how a small area element $du\,dv$ is stretched or compressed when mapped to the xy-plane. So, the change of variables formula becomes

$$\iint_R f(x, y)\,dx\,dy = \iint_S f(g(u, v), h(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\,dv.$$

16.4.1. Example Area Element

Let us return to our earlier example, where $u = xy$, and $v = \frac{y}{x}$. To apply the change of variables, we solve for x and y in terms of u and v. Since $y = vx$, we substitute into $u = xy$ to get

$$u = x(vx) = vx^2 \quad \Rightarrow \quad x = \sqrt{\frac{u}{v}}, \quad y = v\sqrt{\frac{u}{v}} = \sqrt{uv}.$$

We restrict to the first quadrant, so $x, y > 0$.

Now, to compute the transformed integral, we must also calculate the Jacobian determinant $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$. we differentiate:

$$\frac{\partial x}{\partial u} = \frac{1}{2\sqrt{uv}} \qquad \frac{\partial x}{\partial v} = \frac{-1}{2v}\sqrt{\frac{u}{v}}$$

$$\frac{\partial y}{\partial u} = \frac{1}{2}\sqrt{\frac{v}{u}} \qquad \frac{\partial y}{\partial v} = \frac{1}{2}\sqrt{\frac{u}{v}}$$

Now we compute the determinant:

$$\begin{aligned}
\left|\frac{\partial(x,y)}{\partial(u,v)}\right| &= \begin{vmatrix} \frac{1}{2\sqrt{uv}} & -\frac{1}{2}\cdot\frac{\sqrt{u}}{v^{3/2}} \\ \frac{1}{2}\cdot\frac{\sqrt{v}}{\sqrt{u}} & \frac{1}{2}\cdot\frac{\sqrt{u}}{\sqrt{v}} \end{vmatrix} \\
&= \left(\frac{1}{2\sqrt{uv}}\cdot\frac{1}{2}\cdot\frac{\sqrt{u}}{\sqrt{v}}\right) - \left(-\frac{1}{2}\cdot\frac{\sqrt{u}}{v^{3/2}}\cdot\frac{1}{2}\cdot\frac{\sqrt{v}}{\sqrt{u}}\right) \\
&= \frac{1}{4}\cdot\frac{\sqrt{u}}{\sqrt{uv}\sqrt{v}} + \frac{1}{4}\cdot\frac{\sqrt{u}\sqrt{v}}{v^{3/2}\sqrt{u}} \\
&= \frac{1}{4}\cdot\frac{1}{v\sqrt{u}} + \frac{1}{4}\cdot\frac{1}{v} \\
&= \frac{1}{4v}\left(\frac{1}{\sqrt{u}} + 1\right).
\end{aligned}$$

This tells us our new dA:

$$dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \frac{1}{4v}\left(\frac{1}{\sqrt{u}} + 1\right) du\, dv$$

We can assemble all of this into a full example

Example 16.2 (A Change of Variables). To compute the integral $\iint_R x^2\, dA$ over the region R defined by $y = x$, $y = 2x$, $xy = 1$, and $xy = 2$, we make a coordinate substitution to simplify the bounds. Setting $u = xy$ and $v = \frac{y}{x}$, this converts our bounds to a region S in the uv plane:

$$S = \{(u,v) : 1 \leq u \leq 2, 1 \leq v \leq 2\}$$

So, our new bounds are constants, but to convert x, y to u, v we need to *solve for u, v in our coordinate change. This gives $x = \sqrt{\frac{u}{v}}$ and $y = \sqrt{uv}$. Thus, our function to integrate is $x^2 = \left(\sqrt{\frac{u}{v}}\right)^2 = \frac{u}{v}$.

We can also use this to find the area element $dA = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \frac{1}{4v}\left(\frac{1}{\sqrt{u}} + 1\right) du\, dv$. All together then

$$\iint_R x^2 dxdy = \int_1^2 \int_1^2 \frac{u}{v}\frac{1}{4v}\left(\frac{1}{\sqrt{u}} + 1\right) du\, dv$$

Simplifying a bit, this integral is easy as its a function of u times a function of v:

$$= \frac{1}{4}\int_1^2 \int_1^2 \frac{1}{v^2}\left(\sqrt{u} + u\right) dudv$$

16.5. Video Resources

Polar Coordinates

https://youtu.be/51v2UBsO6XY?si=eB0btj5HfONR5pYJ

https://youtu.be/U-13q74uvTo?si=HfIabBbS4UaiAYCF

Cylindrical Coordinates

https://youtu.be/LdqBDbfFKGg?si=7mP72fwj3Ax1D0L0

Spherical Coordinates

https://youtu.be/Zc8uCT-e5KI?si=JO-81C_7Qt60I0TQ

https://youtu.be/LHbpHPoX_Ns?si=6oBo1S4wAoT-MPBB

https://youtu.be/zQJ-XACJd2k?si=rzPjLkUFdGtmjI7v

Examples in Spherical Coordinates:

https://youtu.be/Zc3Ze6qW_Og?si=G0rVjxWbmYKXfZg3

https://youtu.be/jBcNOkwiS6k?si=MV-C12ELGIT6Ux6V

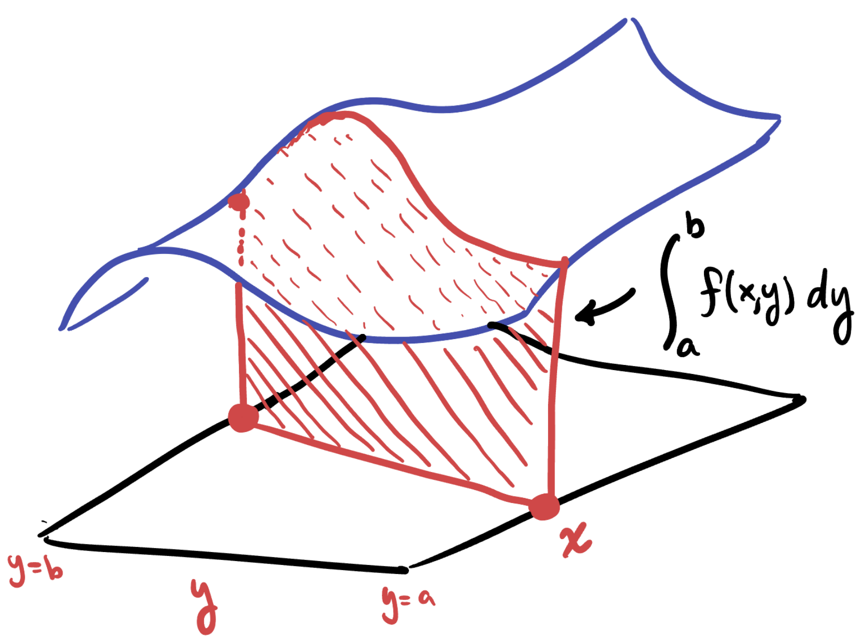
< youtube jBcNOkwiS6k?si=MV-C12ELGIT6Ux6V >

17. Line & Surface Integrals

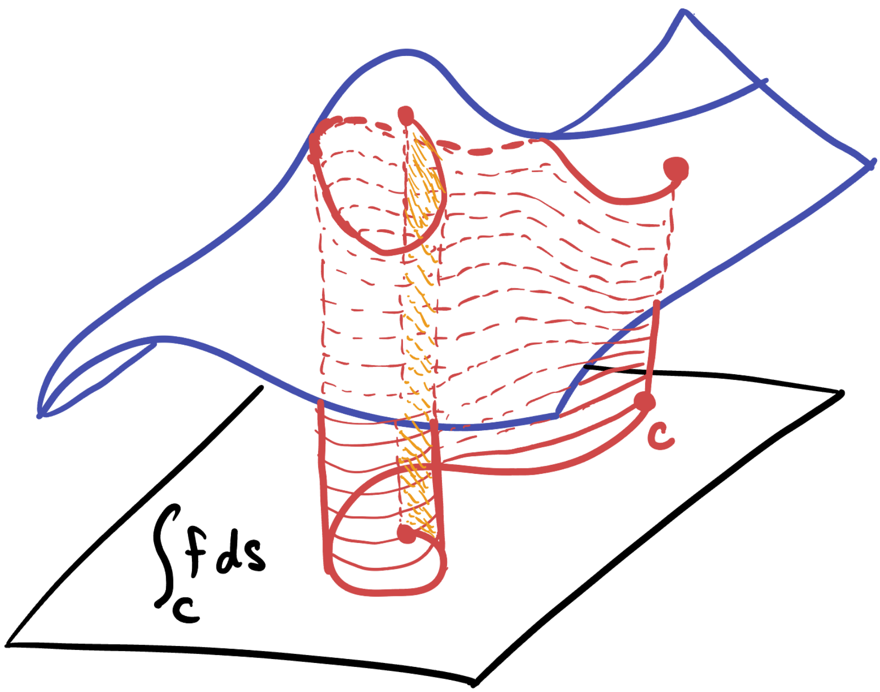
We've learned so far to integrate multivariate functions over a line or over regions in $\mathbb{R}^2$ or $\mathbb{R}^3$. Here we extend our knowledge to consider integrals over a *curve* or a *surface* in space. Such integrals appear in many mathematical and physical contexts, and in particular turn out to be rather important in our final unit, dealing with vector fields.

17.1. Line Integrals

A better name for this object would be a *curve integral* as the goal is to allow the integration of a multivariable function f over some general curve C. We already know how to do this if C is a straight line parallel to the x or y axes, as this is just a slice, representing the *net area above that slice*, or *the total amount of f on that slice*:



This picture caries over directly when C is an arbitrary curve. The area under f above the curve C (or the total amount of f along C, depending on your interpretation) is the *line integral over* C

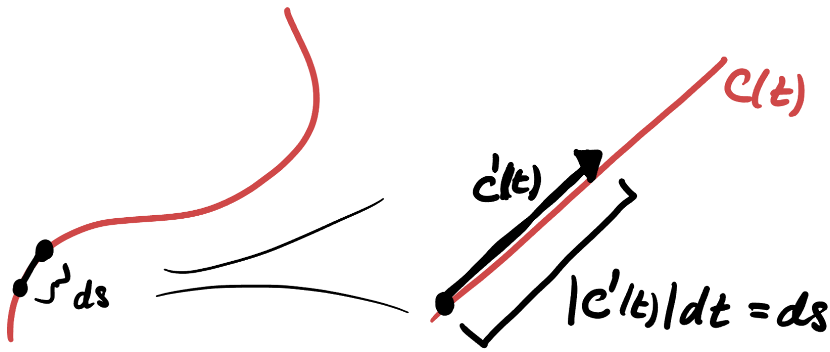


We can denote the domain C as a subscript just like we do for double and triple integrals, and write

$$\int_C f ds$$

Where ds represents an infinitesimal bit of arclength along the curve. When C is a *closed curve* one may optionally modify the integral sign to denote this, writing $\oint_C f\,ds$. Our first goal is to try and figure out how to *compute* this in terms of integrals we know how to do. First: recall that we can represent a curve C by *parameterizing it*, writing it as the image of a function $c(t) = (x(t), y(t))$ in the plane, or $c(t) = (x(t), y(t), z(t))$ in 3 dimensions. We saw back in the chapter on curves how to express a small bit of arclength along a parametric curve:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{\frac{dx^2}{dt^2}dt^2 + \frac{dy^2}{dt^2}dt^2} = \sqrt{x'(t)^2 + y'(t)^2}dt = \|c'(t)\|dt$$



To evaluate a function $f(x, y, z)$ along the curve $c(t) = (x(t), y(t))$, we simply plug the curve into the function. This gives a concrete quantity to integrate:

$$\int_C f\, ds = \int_a^b f(c(t))\|c'(t)\|dt$$

Note the *simplest* line integrals are just when $f = 1$. This is the integral just adding up the infinitesimal arclengths $\int_C ds = \int \|c'\|dt$. We already met this integral long ago - this gives the *arclength* of the curve!

Example 17.1. The slices we have been computing are special cases of line integrals: for a fixed $x = a$ we can parameterize a slice in the y direction by the curve $c(t) = (x, t)$ which has velocity $c' = (0, 1)$ and speed $\|c'\| = 1$, so arclength $ds = \|c'\|dy = 1dy$. Plugging our curve into the function yields $f(c(t)) = f(x, t)$ and thus an integral

$$\int f(x, t)dt$$

The variable has a different name (because we decided to parameterize our curve with t) but this is nothing other than the integral in the y direction, holding x fixed!

Example 17.2. Find $\int_C xyds$ for C the diagonal of the unit square going from $(0, 0)$ to $(1, 1)$.

Example 17.3. Evaluate $\int_C (2 + x^2y)ds$ for C the top half of the unit circle parameterized counterclockwise.

Example 17.4. Evaluate $\oint_C x^2 ds$ for C the circle of radius 2, traversed clockwise.

This tells us what to do anytime we can write C as a differentiable curve: but what if we can't? Sometimes a curve C might be *piecewise*, and has a corner where the edges join up. In this case, we evaluate the line integral by doing each segment of the curve separately, and adding the results:

$$\int_{C_1 \cup C_2} f ds = \int_{C_1} f ds + \int_{C_2} f ds$$

We just have to be sure to parameterize the two curves so that C_1 ends where C_2 starts, so that together they continuously trace the entire curve.

Example 17.5. Find the line integral of x along the curve C which traces out the triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$, starting at the origin and going clockwise.

We break this into three curves:

$$c_1(t) = (t, 0) \qquad c_2(t) = (1-t, t) \qquad c_3(t) = (0, 1-t)$$

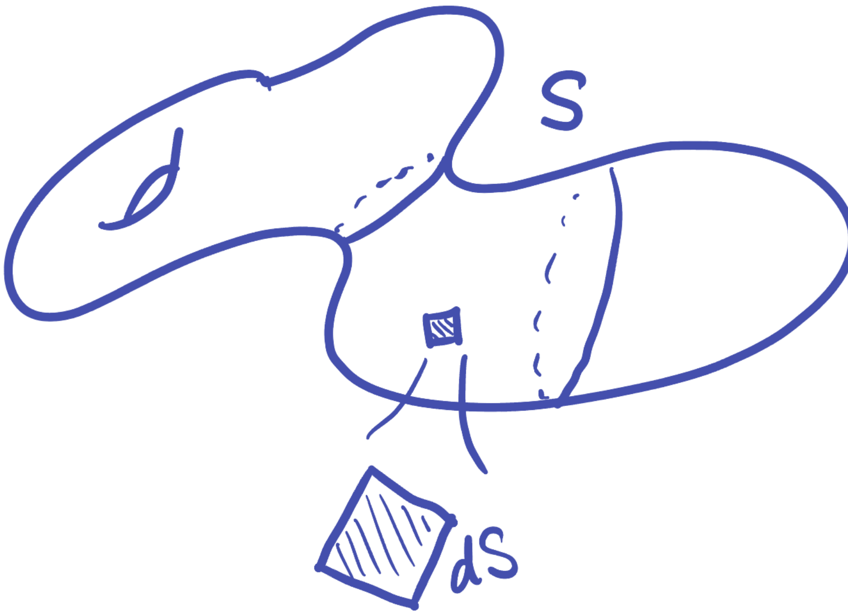
Each is segment is traced from $t = 0$ to $t = 1$. We then just compute the line integral along each and add up the results.

Its no difficulty to generalize this to doing integrals along space curves instead of in the plane:

Example 17.6. Compute the line integral $\int_C f\,ds$ for $f(x,y,z) = y \sin z$ and C the helix $x = \cos t$, $y = \sin t$ and $z = t$ for $t \in [0, 2\pi]$.

17.2. Surface Integrals

We are now interested in extending this notion to integrals over a *surface* instead of a curve. If that surface S is a 2D region inside of the xy plane, we already know how to do this: its just the double integral $\iint_S f\,dA$, where dA is an infinitesimal piece of area on the plane. So, what we are interested in here instead is when S is a surface in 3 dimensional space and $f(x,y,z)$ is a scalar function on 3 dimensional space



These integrals will add up the *total amount of f which lies on the surface S*. This is useful for many things: if f is a density this gives the mass of the object. If f is a *charge density* this would give the total charge on the surface: a computation that is useful inside of batteries, where the anode and cathode may be tightly coiled surfaces. These integrals can also be used to help find the *average value* of a function: dividing the surface integral of f over S by the surface area of S. We will denote such an integral as

$$\iint_S f\,dS$$

Where we think of dS as an infinitesimal piece of surface area. A lot of the work in computing surface integrals is just finding the right form for dS to use, to convert int into a double integral we know how to do. So we begin by looking at some special cases: we actually know how to write the infinitesimal areas on a cylinder or sphere from our work with coordinates!

The Surface of a Cylinder

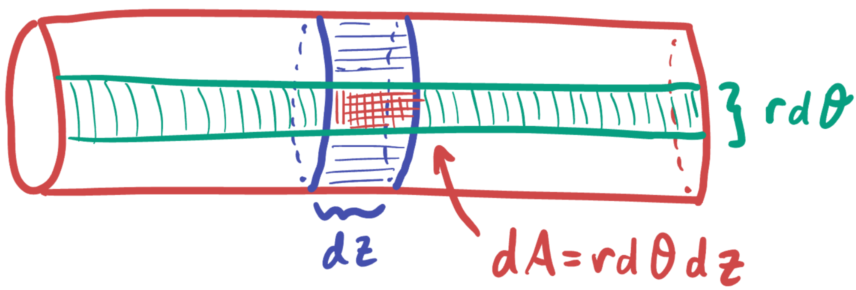
In cylindrical coordinates we convert two of x, y, z to polar r, θ and leave the other alone. The resulting volume element was

$$dV = dzrdrd\theta$$

We can use this to find the *area* element on a cylinder, realizing this simply means that r is held fixed and z, θ are allowed to vary:

Definition 17.1. The area element along a cylinder of fixed radius r is

$$dS = r d\theta dz$$



Thus, to compute a surface integral of f on a portion of a cylinder, we can convert f to cylindrical coordinates and then use the cylinders's area element:

Example 17.7. Compute $\iint_S x^3 dS\$$ for S the cylinder of radius 2 centered on the z-axis between $z = 1$ and $z = 3$.

Here $x = 2\cos\theta$ $y = 2\sin\theta$ and z remains the same, giving $dS = 2d\theta dz$ and $x^3 = (2\cos\theta)^3 = 8\cos^3\theta$. The bounds on the cylinder are $0 \leq \theta \leq 2\pi$ and $z \in [1, 3]$. Putting this all together gives

$$\iint_S x^3 dS = \int_1^3 \int_0^{2\pi} 8\cos^3\theta d\theta dz$$

This is a double integral we can easily evaluate, using the trick $\cos^3\theta = (1 - \sin^2\theta)\cos\theta$ and a u-substitution.

Example 17.8. Compute $\iint_S x dS$ for S the cylinder of radius 1 centered on the x axis from $x = 0$ to $x = 2$.

Here the cylinder is along the x axis so we instead have y, z being the polar coordinates. The cylinders bounds are $\theta \in [0, 2\pi]$ and $x \in [0, 2]$, with $dS = d\theta dx$ since $r = 1$. Plugging all this in yields

$$\iint_S x dS = \int_0^2 \int_0^{2\pi} x dx d\theta$$

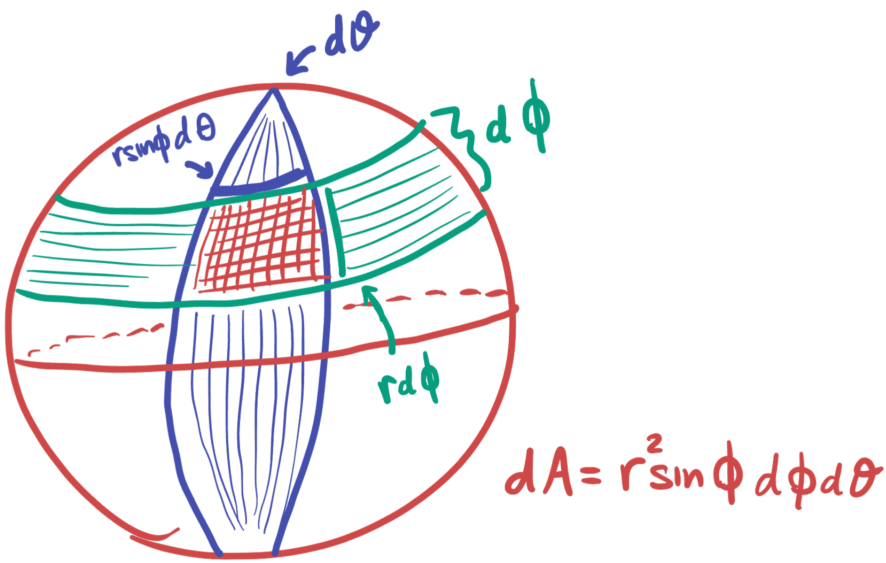
The Surface of a Sphere On a sphere we already have *spherical coordinates* for which we computed a nice volume element

$$dV = \rho^2 \sin\phi d\rho d\phi d\theta$$

If we wish to look at the *area of a sphere* we just need to fix ρ to some constant value (its radius), and then consider only small changes in ϕ and θ (we are not changing ρ as we wish to stay on the surface!)

Definition 17.2. The infinitesimal area element on a sphere of radius r is

$$dS = r^2 \sin\phi d\phi d\theta$$



Thus, to compute a surface integral of f on a portion of a sphere, we can convert f to spherical coordinates and then use the sphere's area element:

Example 17.9. Compute $\iint_S z dS$ for S the upper hemisphere of the sphere of radius 2.

In spherical coordinates $z = \rho \cos\phi$, and here the radius is fixed at 2 so $z = 2\cos\phi$. The spherical coordinate bounds for the upper hemisphere have $\phi \in [0, \pi/2]$ and $\theta \in [0, 2\pi]$, so plugging these in with the area element $dS = 4\sin\phi d\phi d\theta$ yields

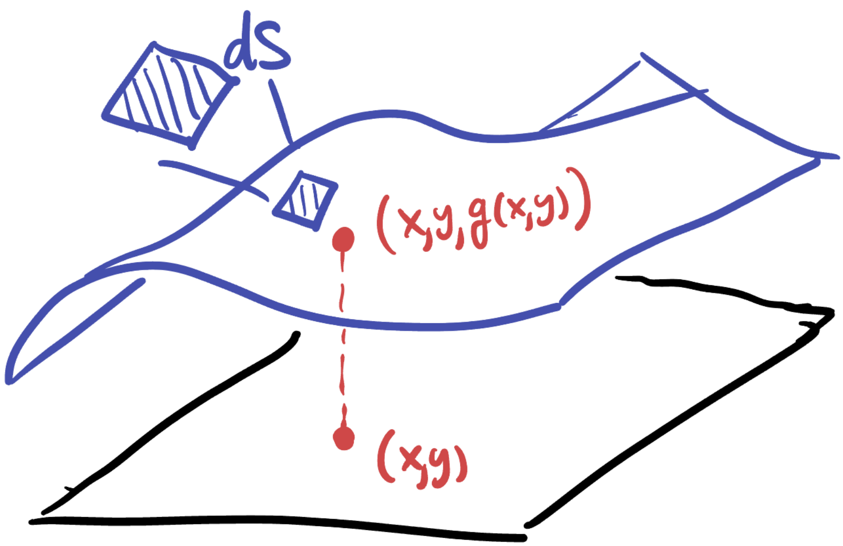
$$\iint_S z dS = \int_0^{2\pi} \int_0^{\pi/2} (2\cos\phi)(4\sin\phi) d\phi d\theta = 8 \int_0^{2\pi} \int_0^{\pi/2} \cos\phi \sin\phi d\phi d\theta$$

This integral is easily evaluated with a u-sub $u = \sin\phi$.

Example 17.10. Compute $\iint_S (x^2 + y^2 + z^2) dS$ for S the sphere of radius 3 centered at the origin.

The Graph of a Function

What about when our surface *isnt* something simple like this that we already know the answer to? Let's start with the case that our surface is the graph of a function $z = g(x, y)$.



How do we describe an infinitesimal area element on this surface? Recalling back to our earlier study of functions and their graphs, we can easily find two *tangent vectors* to our graph by writing it as a parametric surface

$$(x, y, g(x, y))$$

and then taking the partial derivatives. This gives a vector v_x in the x direction and v_y in the y direction

$$v_x = \frac{\partial}{\partial x}(x, y, g(x, y)) = \langle 1, 0, g_x \rangle \qquad v_y = \frac{\partial}{\partial y}(x, y, g(x, y)) = \langle 1, 0, g_y \rangle$$

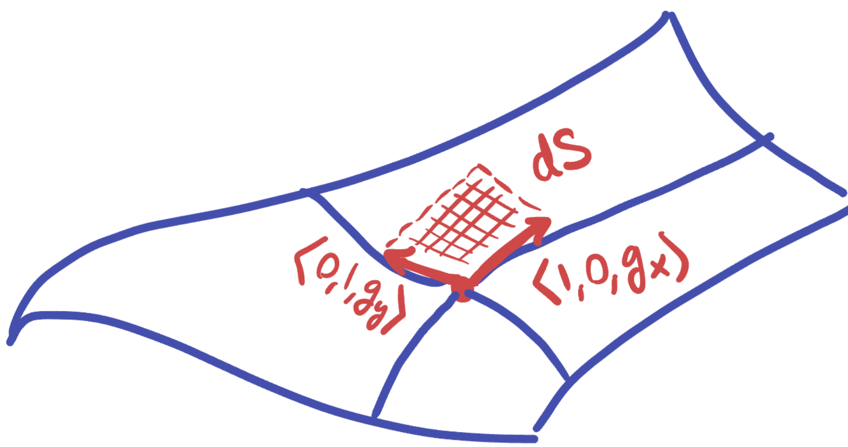
These two vectors span an infinitesimal parallelogram, whose area is the area element we seek. How do we find the area of a small parallelogram again? The cross product! Thus, the area element is simply

$$dS = \|v_x \times v_y\| dxdy$$

Computing this cross product and taking the magnitude gives our answer

$$v_x \times v_y = \begin{vmatrix} i & j & k \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = i(-g_x) - j(g_y) + k(1) = \langle -g_x, g_y, 1 \rangle$$

$$\|v_x \times v_y\| = \sqrt{(-g_x)^2 + (-g_y)^2 + 1^2} = \sqrt{1 + g_x^2 + g_y^2}$$



Definition 17.3. The area element along the graph of a function $z = g(x, y)$ is given by

$$dS = \sqrt{1 + g_x^2 + g_y^2}\, dxdy$$

This gives us a means to evaluate surface integrals along such objects: we simply evaluate f at points along the graph, and integrate with respect to this dS.

Example 17.11. Find the area of the paraboloid $z = x^2 + y^2$ that lies below the plane $z = 9$.

Example 17.12. Find $\iint_S ydS$ for S the surface $z = x + y^2$ for $0 \leq x \leq 1$ and $0 \leq y \leq 2$.

Ans: $\frac{13\sqrt{2}}{3}$.

Just like for Line Integrals, surface integrals can be computed over regions bounded by two or more surfaces joined together by computing each part separately and adding the results:

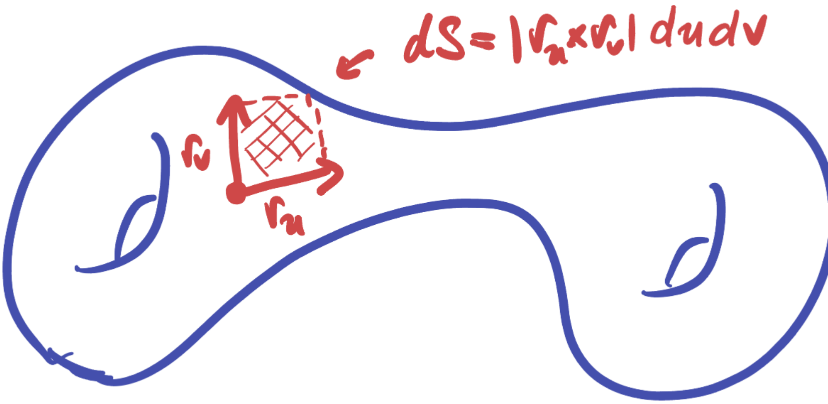
Example 17.13. Let S denote the closed surface defined by $z = 2 - x^2 - y^2$ and the plane $z = 1$. Compute the surface integral $\iint_S x + zdS$$

General Surface Integrals

The cases above actually cover many useful examples, so we will not often need the general case. But we've come so far we might as well spell it out precisely. Let's say we have a surface S in space described by a parametric equation $r(u, v) = (x(u, v), y(u, v), z(u, v))$. Its easy to evaluate a scalar function $f(x, y, z)$ on the surface, we just plug in this parameterization

$$f(r(u, v)) = f(x(u, v), y(u, v), z(u, v))$$

To get an infinitesimal surface area element, we can follow the trick that worked for graphs: if we can find two tangent vectors at a point we can find the area of the infintiesimal parallelogram they span using the *cross product*.



Here the tangent vectors are the u and v derivatives of the parameterization

$$r_u = \frac{\partial r}{\partial u} = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right\rangle \qquad r_v = \frac{\partial r}{\partial v} = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right\rangle$$

The *normal vector* to the surface at this point is the cross product $n = r_u \times r_v$ and its magnitude is the infinitesimal area element:

$$dS = \|r_u \times r_v\| du dv$$

This converts everything into an explicit double integral

$$\iint_S f\, dS = \iint_D f(r(u, v)) \|r_u \times r_v\| du dv$$

Where D is the domain of the parameterization (its u bounds and v bounds).

Example 17.14. Compute $\iint_S xyzdS$ for S the cone parameterized by $x = u\cos v$, $y = u\sin v$ and $z = u$ for $u \in [0,1]$ and $v \in [0,2\pi]$.

Example 17.15. Compute $\iint_S x^2 + y^2\, dS$ for S the surface with parameteric equation $r(u,v) = (2uv, u^2 - v^2, u^2 + v^2)$ where (u,v) lies in the unit disk $u^2 + v^2 \leq 1$.

17.3. Video Resources

17.3.1. Line Integrals

https://www.youtube.com/watch?v=WA5_a3C2iqY&t=710s

https://www.youtube.com/watch?v=fqVEuFldFuA

17.3.2. Surface Integrals

https://youtu.be/ucFLXZmgZ54?si=z7SVquwxYLvaFlrf

https://youtu.be/heU3Ph20QZQ?si=Wy2W-dfUumYV_HuS

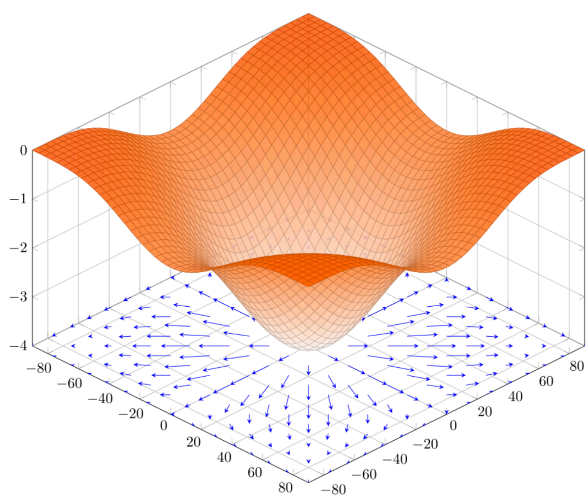
https://youtu.be/jBcNOkwiS6k?si=xFrJgZt59Cd1qVZt

Part V.

Vector Fields

18. Introduction

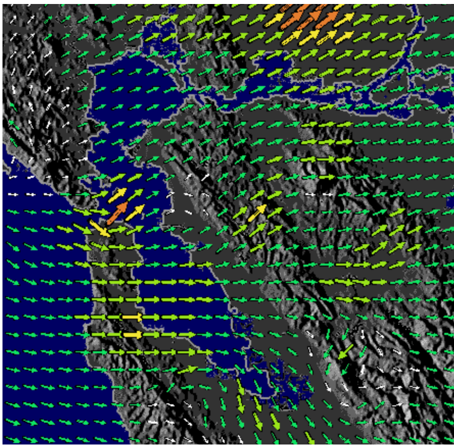
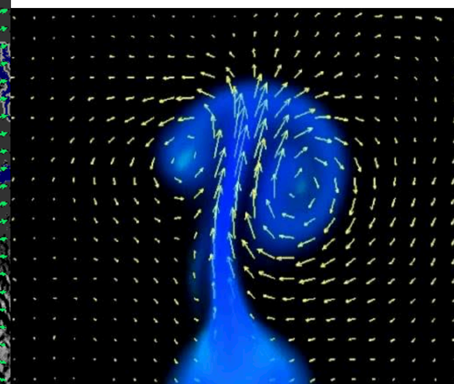
At the end of our course, we will dive into one of the areas in mathematics where all the tools we've developed come together: the calculus of vector fields. A *vector field* is a function which assigns a vector to each point in space. We have mostly been dealing with scalar functions so far, which assign a single number to each point, but we have of course met one very important example of a vector field, *the Gradient*.

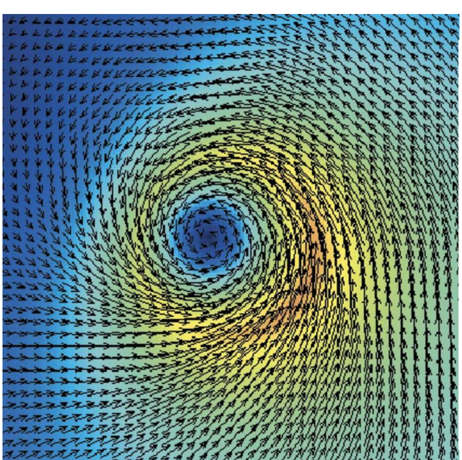
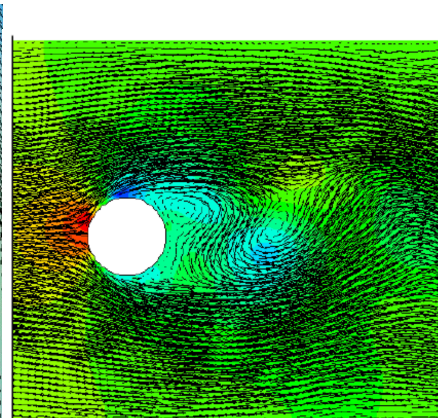


We are very familiar with vector fields from day-to-day life however, for example the wind vector field which assigns each point (x, y) on a map a vector $W(x, y) = \langle f(x, y), g(x, y)\rangle$ depicting the wind's magnitude and direction at that location. Below are two such examples, one being a wind-map over San Francisco and the velocity field of an airflow breaking apart into turbulent vorticies, behavior that it is crucial to understand when engineering aircraft and spaceship reentry.

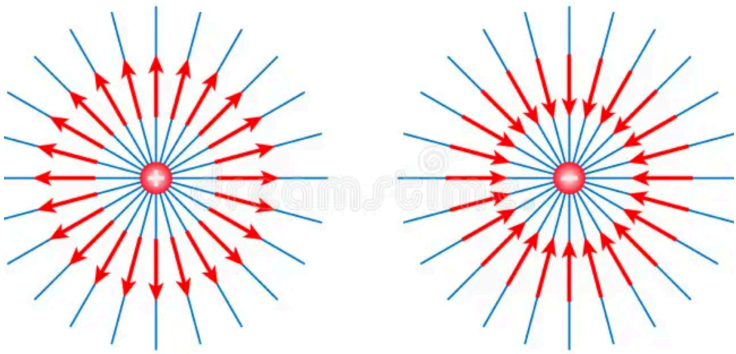
Vector fields are ubiquitous throughout science. Generalizing the wind example above, they play a role in our mathematical modeling of all fluids, where we track a scalar quantity (the pressure) as well as a vector quantity (the velocity) at each point in space. Below the pressure field is shown in color (warmer colors = high pressure, cool colors = low pressure) superimposed with the velocity field drawn as arrows.

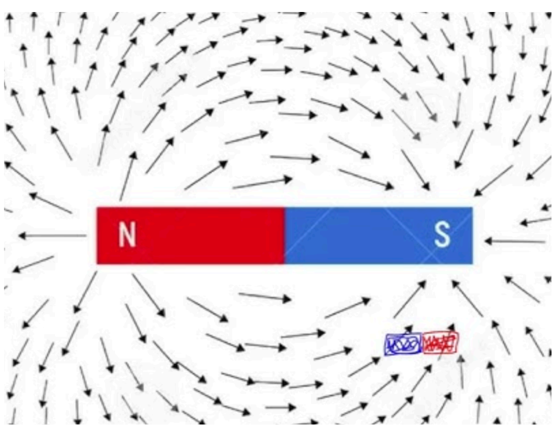
Electricity and magnetism are given by *force fields*, which assign to each point in space a vector from which one can compute the *force felt by an object at that location* (for the Electric field, this force is directly in the direction given by the vector, for the

magnetic field, the force is in the direction of the *cross product* of this vector with the object's velocity)





The force of gravity is also descried by a force field, with vectors pointed in the direction an object is pulled by gravity (more on this below!).

The take away here is that vector fields, just like scalar fields, show up all over in mathematics and the sciences, so we should like to extend our theory of multivariable calculus to include them as well. In this brief introductory section we look at how to write down vector fields and how to plot them; in the following sections we will consider their derivatives (curl and divergence) and their integrals (line and surface integrals of vector fields).

18.1. Working with Vector Fields

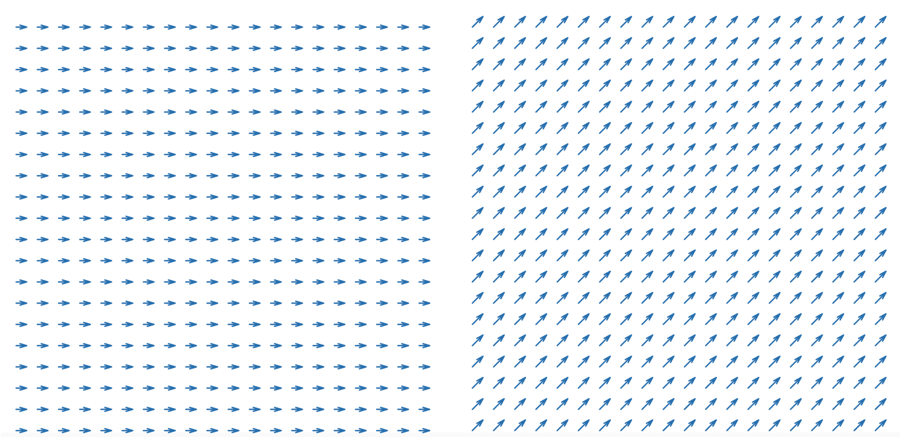
We will write things down for the 2 dimensional case here, but everything we say applies in all higher dimension (and, we will give 3D examples to follow). A 2 dimensional vector has two components $\langle p, q\rangle$. In a 2D vector field, we must assign a 2D vector to each point (x, y) in a 2 dimensional region, so these components are themselves functions of x and y. That is, we can write a field $\vec{V}$ as

$$\vec{V} = \langle P(x, y), Q(x, y) \rangle$$

or, if we wish to use the basis vector notation, P is the $\hat{i}$ component of $\vec{V}$ and Q is the $\hat{j}$ component, so we have

$$\vec{V} = P(x, y)\hat{i} + Q(x, y)\hat{j}$$

And similarly in 3 dimensions where we may write $\vec{V} = \langle P(x, y, z), Q(x, y, z), R(x, y, z) \rangle$ or $\vec{V} = P\hat{i} + Q\hat{j} + R\hat{k}$. If the functions P, Q are constant then the vector field V assigns the same vector to each point in space.



The constant vector fields $\vec{V} = \langle 1, 0 \rangle$ (left) and $\vec{W} = \langle 1, 1, \rangle$ (right).

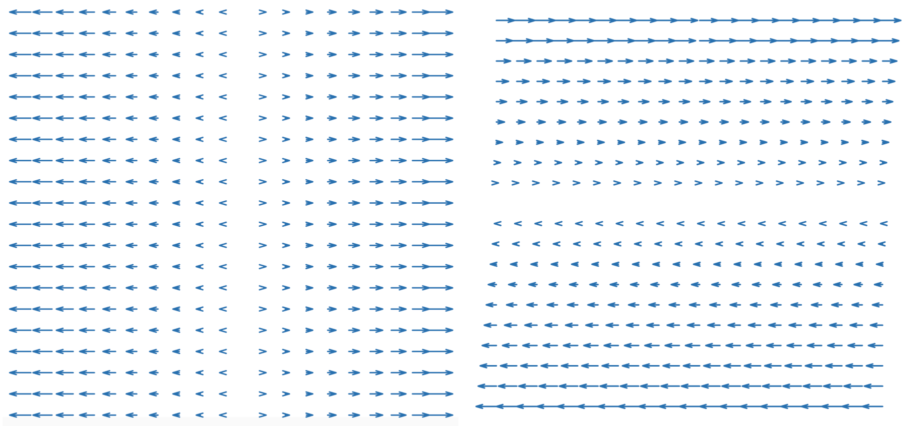
Other vector fields vary in *magnitude* but not direction, by taking a fixed vector and scalar multiplying it by a function of (x, y):

Or, a vector field can vary in *direction* without changing magnitude, such as a field of all unit vectors. Such a vector field can be written down by taking any nonzero vector field and dividing by its magnitude, for example.

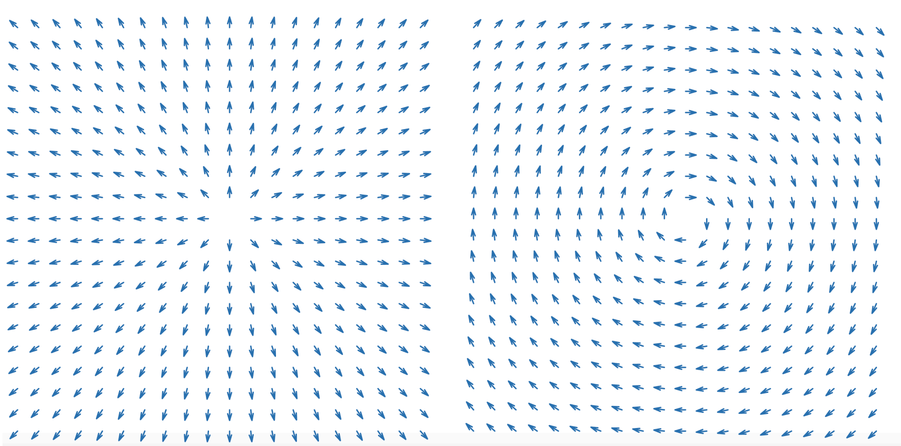
General vector fields vary in both magnitude and direction, like the wind. Here's a quick desmos program that you can use to draw your own

And below is another graphing calculator I wrote that you can use to plot vector fields in the plane.

https://stevejtrettel.site/code/2022/vector-field-2d



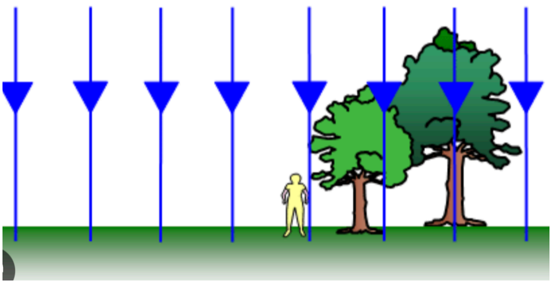
The vector fields $\vec{V} = \langle x, 0 \rangle$ (left) and $\vec{W} = \langle y, 0 \rangle$ (right).



The vector fields $\vec{V} = \langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \rangle$ (left) and $\vec{W} = \langle \frac{y}{\sqrt{x^2+y^2}}, \frac{-x}{\sqrt{x^2+y^2}} \rangle$ (right).

18.1.1. Example: Gravity

Let's try to *use* vector fields to write down the gravitational force. First, let's start with Galileo's approximation that gravity is constant and pointed straight down towards the surface of the earth.



If we model space here as the xy plane, with the y axis upwards and the x axis horizontal, gravity is modeled by a constant vector pointed straight downwards: indeed for some constant g we may write the gravitational *acceleration* as

$$\vec{a} = \langle 0, -g \rangle = -g\hat{j}$$

and then since $F = ma$, the force felt by an object of mass m is just the scalar multiple of this by m:

$$\vec{F} = \langle 0, -mg \rangle = -mg\hat{j}$$

This approximation works perfectly well on the surface of the earth, as we know from mathematical modeling experience in highschool or introductory college physics (where, neglecting air resistance, one finds that objects thrown into the air follow parabolas.) However at the larger scale of planetary dynamics, Newton proposed an updated law, which reads like this in words:

The gravitational acceleration of experienced from an object points in the direction of that object, and has magnitude proportional to the mass of the object divided by the square of the distance that object is away.

Let's convert this into a *formula* for the gravitational field $\vec{F}(x, y, z)$ in 3 dimensional space. To make things a little easier, lets place our object at the origin $(0, 0, 0)$ and say its mass is M. If you are at the location (x, y, z), then the force on you is supposed to *point towards the object*, so it should point in the direcion of the vector connecting you to the origin:

$$\vec{d} = (0, 0, 0) - (x, y, z) = \langle -x, -y, -z \rangle$$

Because we are given information about both the *direction* and *magnitude* in the description, it might be useful to record this direction as a *unit vector* so all the magnitude information can be scalar multipled by it. This gives

$$\hat{d} = \left\langle \frac{-x}{\sqrt{x^2+y^2+z^2}}, \frac{-y}{\sqrt{x^2+y^2+z^2}}, \frac{-z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

The distance to the origin is $\sqrt{x^2+y^2+z^2}$ and the *squared* distance to the origin is $x^2+y^2+z^2$. Thus, the magnitude of the gravitational force is supposed to be proportional to $M/(x^2+y^2+z^2)$. Let's call the proportionality constant G (its value is set by whatever units we are using, in standard SI units its about $6 \cdot 10^{-11}$.) This tells us the magnitude of the gravitational acceleration should be

$$\frac{GM}{x^2+y^2+z^2}$$

Multiplying this by the direction gives the acceleration, and the force is given by then further multiplying by the mass of you (the object at (x, y, z)): let's call that m

$$\vec{F} = \frac{GMm}{x^2+y^2+z^2} \left\langle \frac{-x}{\sqrt{x^2+y^2+z^2}}, \frac{-y}{\sqrt{x^2+y^2+z^2}}, \frac{-z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

19. Circulation and Flux

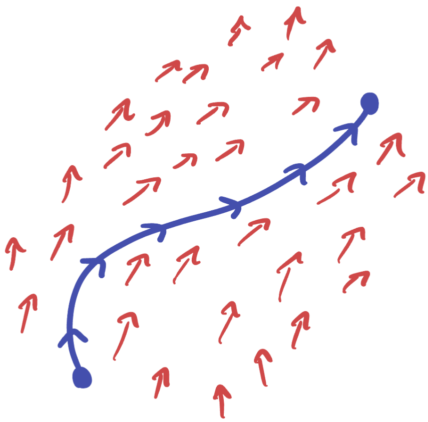
Having an introduction to vector fields and their derivatives, we turn to their integrals. Many real world quantities arise as the integrals of vector fields, for example:

- Did the wind outside improve or hurt my time running a race?
- How does a magnetic field circulate around a wire?
- How much water flowed through my fishing net as I was trawling?
- How much power is available in the wind passing through a windmill?
- How much sunlight is hitting a solar panel, when the sun is at an angle in the sky?
- Where did a collision occur in a particle detector, given the spray of particles we see?

These sorts of questions essentially fall into two different categories: one is trying to measure how much a vector fields is *pointed along a curve* and the other is measuring how much a vector field *flows through a region. We will refer to the first of these as the* circulation* of a vector field, and the second as its *flux.*

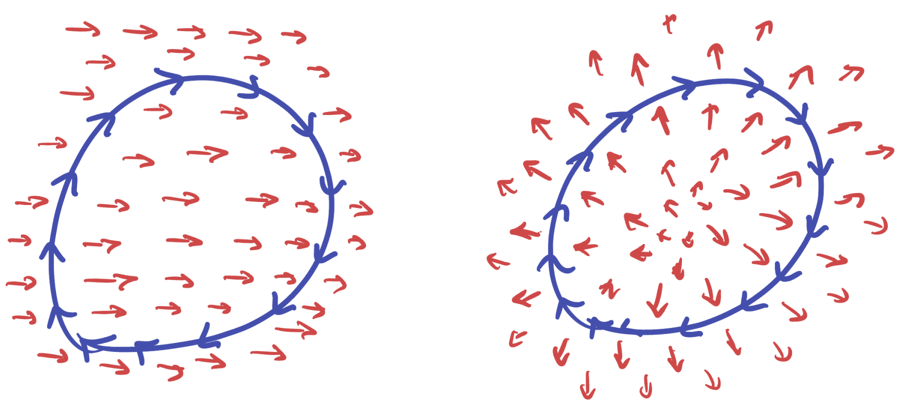
19.1. Circulation

Circulation is defined given an *oriented curve* (that is, a curve with a specified direction for its parameterization) and a *vector field.* The goal is to measure whether the vector field's net affect is to *point along the direction of motion* for the curve, or to *point against the motion of the curve.* We will call the former *positive circulation* and the latter *negative circulation.*



The left vector field has *positive circulation* along the curve, whereas the right has *negative*.

A vector field has zero circulation along a curve if it overall neither aligns or anti-aligns on average. This can happen in several ways: perhaps it lines up with the curve half the time and opposes it the other half, or alternatively could simply be *perpendicular to the curve* at all times.

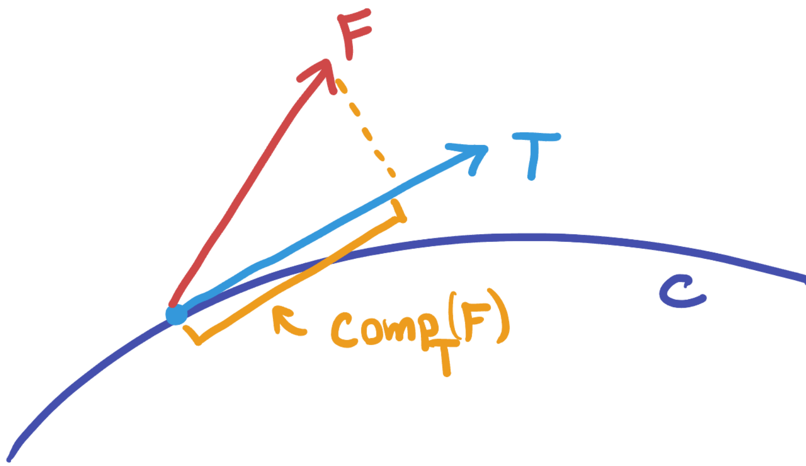


These two vector fields have *zero circulation* around the curve.

We need to do some work now to turn this definition into a precise mathematical quantity. Since we are measuring a *net affect along a curve*, we can greatly simplify things by breaking down into two steps

- (1) Quantify how well a vector field aligns / anti-aligns with a curve at a point.
- (2) Get the net affect by adding these all up in a *line integral* along the curve.

Focusing on point (1), we can zoom in on a single point along our curve p and reaching back deep into our memories from early in the course, notice that we actually already know how to compute the amount of F pointing along our curve! It's just the *component of F in the direction tangent to the curve.



We can compute this by choosing any tangent vector T pointing along the curve (in the direction of motion), and using a dot product

$$\mathrm{comp}_T(F) = \frac{F \cdot T}{|T|}$$

Definition 19.1 (Circulation Integral)**.** The circulation of a vector field F along an oriented curve C is defined as

$$\int_C \mathrm{comp}_T(F)\, ds = \int_C \frac{F \cdot T}{|T|}\, ds$$

where T is a choice of tangent vector to the curve

Now that we have a conceptual formula down, lets try to simplify it into something usable. Given a parameterization $\vec{c}(t)$ of our curve, at time t we can choose the tangent vector to simply be the derivative $\vec{c}'(t)$, and the vector field at that point is simply $\vec{F}(\vec{c}(t))$. This gives an explicit formula for the alignment at a point

$$\frac{\vec{F}(\vec{c}(t)) \cdot \vec{c}'(t)}{|\vec{c}'(t)|}$$

This simplifes greatly inside of a line integral, because the definition of infinitesimal arclength also involves $|\vec{c}'|$:

$$\int_C \frac{\vec{F}(\vec{c}(t)) \cdot \vec{c}'(t)}{|\vec{c}'(t)|} ds = \int \frac{\vec{F}(\vec{c}(t)) \cdot \vec{c}'(t)}{|\vec{c}'(t)|} |\vec{c}'(t)| dt = \int \vec{F}(\vec{c}(t)) \cdot \vec{c}'(t)\, dt$$

The norms cancel and we end up with a relatively simple integral: just the dot product of F along the curve with the curve's derivative! This form makes it easy to see *how* to compute such an integral so we record it for (much) future use as a theorem

Theorem 19.1 (Circulation Integral, in Usable Form). *Given a vector field F and parametric curve $c(t)$ for $t \in [a,b]$, the circulation integral can be computed via*

$$\int_C \mathrm{comp}_T(F)\, ds = \int_a^b F(c(t)) \cdot c'(t)\, dt$$

Let's do a quick example to confirm this is an easy computation:

Example 19.1. Compute the circulation of $F = \langle xy, y+1 \rangle$ along the parabola $c(t) = (t, t^2)$ for $t \in [0,1]$.

We compute the vector field along the curve by simply subbing in the curves $x = t$, $y = t^2$ for x, y:

$$F(c(t)) = F(t, t^2) = \langle (t)(t^2), (t^2) + 1 \rangle = \langle t^3, t^2 + 1 \rangle$$

Differentiating c gives the tangent vector

$$c'(t) = \langle 1, 2t \rangle$$

computing their dot product yields the integrand

$$F(c(t)) \cdot c'(t) = \langle t^3, t^2 + 1 \rangle \cdot \langle 1, 2t \rangle = t^3 + (t^2 + 1)2t = 3t^3 + 2t$$

Integrating over the bounds gives the net circulation

$$\int_0^1 3t^3 + 2t\, dt = \frac{3t^4}{4} + t^2 \Big|_0^1 = \frac{3}{4} + 1 = \frac{7}{4}$$

If the curve C consists of multiple piecewise components, one simply repeats this procedure for each component and adds the results, exactly as for a standard line integral.

Like many things in multivariable calculus, there are multiple notations for the circulation integral, suited to different purposes. In addition to the form we have right now $\int_a^b F(c(t)) \cdot c'(t)\, dt$, there is also a longer and a shorter notation in use. First, for the shorter one we can adopt the shorthand of $d\vec{s}$ for the *infinitesimial displacement vector*

$$d\vec{s} = \vec{c}'(t) dt$$

much as we use $ds = |\vec{c}'(t)|dt$ for infinitesimal distance. Substituting this into our integral gives the notation

$$\int_C \vec{F} \cdot d\vec{s}$$

Which is easy to unpack directly back to our other notation by substituting in the definition of $d\vec{s}$, and its conciseness is particularly helpful when doing long theoretical calculations where we aren't necessarily immediately trying to compute the integral. On the other hand, one might also wish for an *even more expanded notation* than our original, that is amenable to *directly plugging in for computation.* One can come up with such a notation by expanding F as $F = \langle P, Q \rangle$ and writing out the dot product:

$$F \cdot c' = \langle P, Q \rangle \cdot \langle x', y' \rangle = Px' + Qy'$$

This is our integrand, and so we can rewrite the circulation as

$$\int_C (Px' + Qy') = \int Px'dt + Qy'dt$$

using the shorthand that $x'dt = dx$ and $y'dt = dy$, this yields the memorable notation

$$\int_C \vec{F} \cdot d\vec{s} = \int_C Pdx + Qdy$$

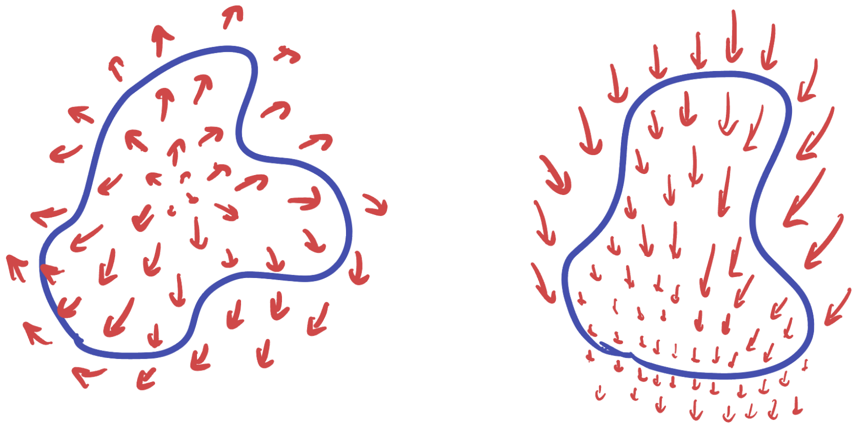
We collect all these notations below in one place for future reference.

Definition 19.2 (Notation for Circulation Integrals)**.** Given an oriented curve c and a vector field F we have several different notations for the circulation integral of F along c:

$$\text{Circulation} = \int_C \text{comp}_{\vec{T}}(\vec{F})\, ds = \int_C \vec{F} \cdot d\vec{s} = \int_C Pdx + Qdy$$

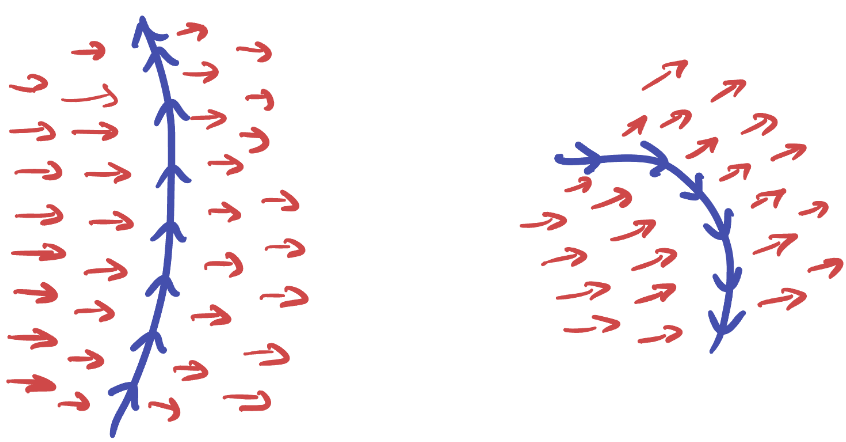
19.2. Flux

The story of *flux through a curve* is very similar to circulation, save one crucial detail. We are no longer interested in flow *along* the curve but rather flow *through*. We make the convention that if a curve is closed (which will be our most common use case) *positive* means net flow *outwards* and negative is flow *inwards*



Positive flux (left) as the vector field is flowing out of the curve at all points. Negative flux (right) even though the vector field is flowing overall downwards (into the curve at the top and out at the bottom), the inflow is greater than the outflow so the *net flux* is inwards.

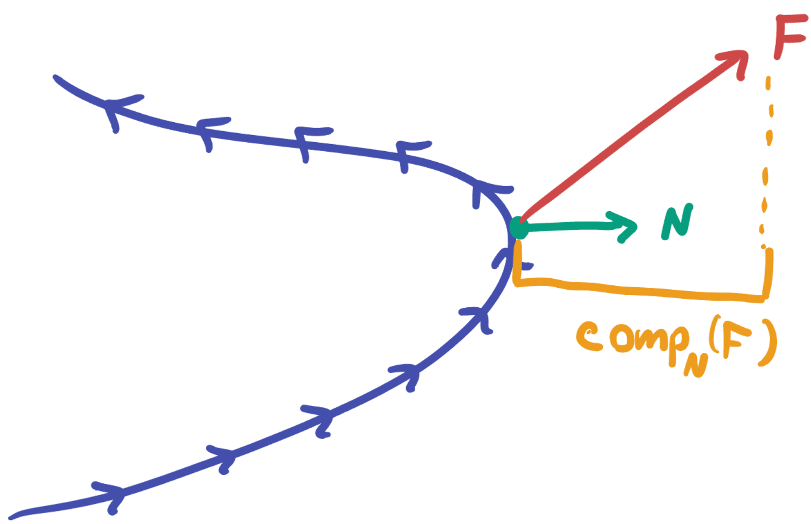
And for an otherwise arbitrary parameterized curve, we will call positive flow *from left to right* along the direction of motion, and negative a flow from right to left (this matches with the above when a closed curve is parameterized counterclockwise - the math standard orientation).



The sign of flux has to do with the *relative orientation* of the curve and vector field. The left shows positive flux and the right negative flux, even though the vector field in both cases goes left to right.

Again its clear the net flow is going to be a line integral of the flow through a point, and so we need only compute it locally. Here sits the only change from our previous case: to measure *outflow* we are now concerned about the projection onto the normal instead of the tangent:

$$\mathrm{comp}_N(F) = \frac{F \cdot N}{|N|}$$

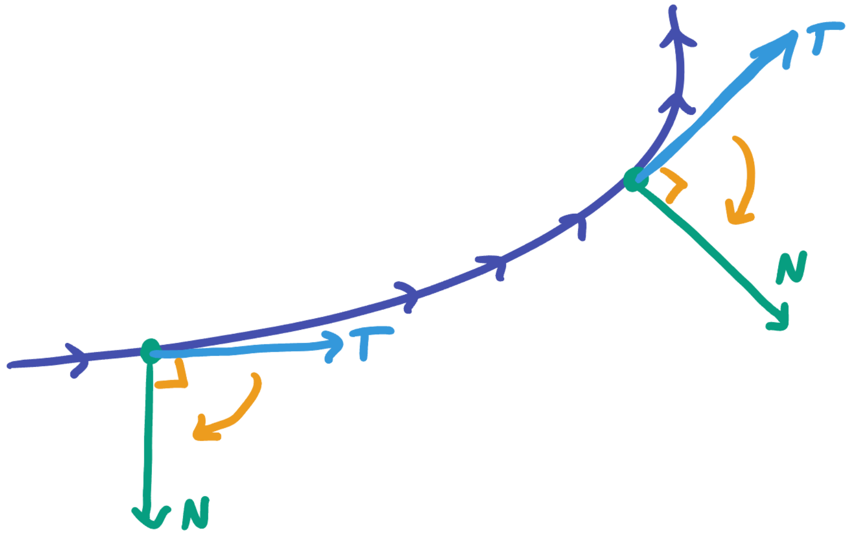


Definition 19.3 (Flux Integral)**.** The flux of a vector field F through an oriented curve C is defined as

$$\int_C \mathrm{comp}_{\vec{N}}(\vec{F})\, ds = \int_C \frac{\vec{F} \cdot \vec{N}}{|\vec{N}|}\, ds$$

where N is a choice of normal vector to the curve at each point

We again try to unpack this definition using an actual parameterization of the curve $c(t) = (x(t), y(t))$. Differentiation again gives us direct access to a *tangent vector*, and to get a correctly oriented *normal vector* we simply need to rotate this vector by $\pi/2$ clockwise:



How much of F passes *through* the curve is quantified by the component of F in the *normal* direction.

Such a rotation is accomplished by the transformation $(x, y) \mapsto (y, -x)$, which takes the tangent vector c' to the normal vector $n = \langle y', -x' \rangle$. Using this normal vector in our calculation, we find

$$\mathrm{comp}_n(F) = \frac{F \cdot \langle y', -x' \rangle}{|\langle y', -x' \rangle|}$$

This also simplifies greatly inside of the line integral, since the length of n and the length of c' are equal to one another (one is just a rotation of the other, after all):

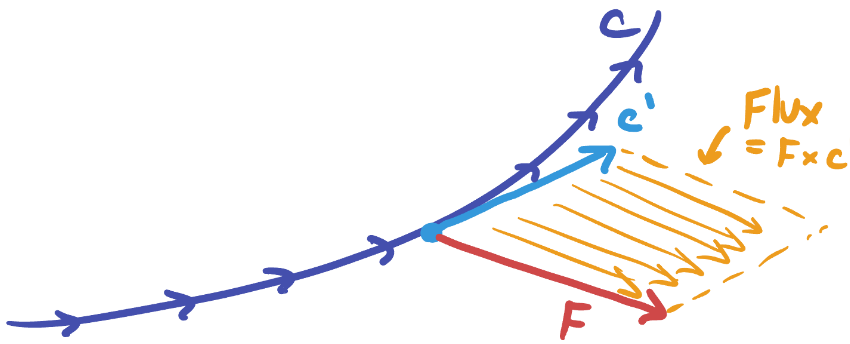
$$\begin{aligned} \int_C \frac{F \cdot \langle y', -x' \rangle}{|\langle y', -x' \rangle|} ds &= \int_C \frac{F \cdot \langle y', -x' \rangle}{|\langle y', -x' \rangle|} |\langle x', y' \rangle| \, dt \\ &= \int_C F \cdot \langle y', -x' \rangle \, dt \end{aligned}$$

Actually performing this dot product for $F = \langle P, Q \rangle$ suggests a further simplification that will make the formula even more memorable:

$$\langle P, Q \rangle \cdot \langle y', -x' \rangle = Py' - Qx'$$

This is itself just a *cross product*!

$$Py' - Qx' = \begin{vmatrix} P & Q \\ x' & y' \end{vmatrix} = \langle P, Q\rangle \times \langle x', y'\rangle = F \times c'$$



This has a nice interpretation: this parallelogram is made of many parallel copies of F infinitesimally translated along c': so its area truly is capturing *the amount of F flowing through c along this segment of arc.*

In addition to being a memorable picture, this new formula lends itself directly to computation, so let's record it in a box and do an example

Theorem 19.2. *Given a vector field F and parametric curve $c(t)$ for $t \in [a, b]$, the flux integral can be computed via*

$$\int_C \mathrm{comp}_N(F)\,ds = \int_a^b F(c(t)) \times c'(t)\,dt$$

Example 19.2. Compute the flux of $F = \langle x + y, y\rangle$ through the unit circle (oriented counterclockwise).

We start by writing down this standard parameterization $c(t) = (\cos t, \sin t)$ for $t \in [0, 2\pi]$. Then compute $c'(t) = \langle -\sin(t), \cos(t)\rangle$ and F along the curve

$$F(c(t)) = F(\cos t, \sin t) = \langle \cos(t) + \sin(t), \sin(t)\rangle$$

Taking the cross product gives the integrand

$$F \times c' = \begin{vmatrix} \cos(t) + \sin(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = (\cos t + \sin t)\cos t + \sin^2 t$$

$$= \cos^2 t + \cos t \sin t + \sin^2 t = 1 + \cos t \sin t$$

Integrating then directly gives the flux

$$\int_0^{2\pi} 1 + \cos(t)\sin(t)\,dt = 2\pi + 0 = 2\pi$$

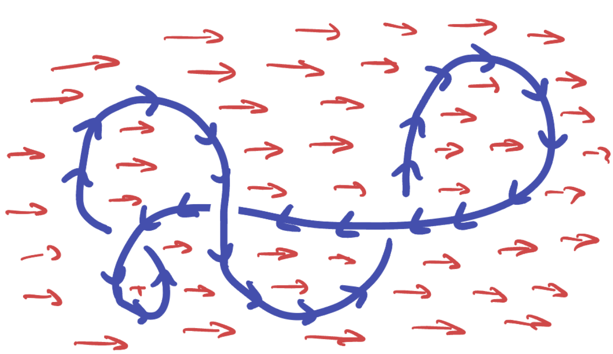
For consistency we can also seek better, more memorable notations for the flux integral like we did for circulation. The integrand $F(c(t)) \times c'(t)dt$ immediately collapses to $\vec{F} \times d\vec{s}$ under our shorthand $d\vec{s} = \vec{c}'(t)dt$, and the longer, 'plug it in form' is deducible from our computations above $Pdy - Qdx$

Definition 19.4 (Notation for Flux Integrals)**.** Given an oriented curve c and a vector field F we have several different notations for the flux integral of F through c:

$$\text{Flux} = \int_C \text{comp}_N(F)\,ds = \int_C \vec{F} \times d\vec{s} = \int_C Pdy - Qdx$$

19.3. Circulation and Flux in 3 Dimensions

The defintiion of circulation remains unchanged in three dimensions, as it still makes perfect sense to try and measure how much a 3D vector field aligns with a 3D curve. Indeed, the formula remains the same, integrating the tangential component along the curve, and the entirety of the previous discussion goes through just with one added coordinate



Definition 19.5 (Circulation in 3D)**.** Given a vector field $\vec{F} = \langle P, Q, R\rangle$ and a parametric curve $c(t) = (x(t), y(t), z(t))$, the net circulation of F along c is given by the integral

$$\int_C \text{comp}_T(F)\,ds = \int_C \vec{F} \cdot d\vec{s} = \int_C Pdx + Qdy + Rdz$$

Example 19.3. Compute the circulation of $F = \langle xy, y+z, z^2\rangle$ along the circle $c(t) = \langle \cos(t), \sin(t), 1\rangle$.

Again we find F along the curve as

$$F(c(t)) = F(\cos t, \sin t, 1) = \langle \cos t \sin t, \sin t + 1, 1^2\rangle$$

and the tangent vector to the curve as

$$c'(t) = \langle -\sin t, \cos t, 0\rangle$$

Their dot product is the integrand

$$\langle \cos t \sin t, \sin t + 1, 1^2\rangle \cdot \langle -\sin t, \cos t, 0\rangle = -\cos t \sin^2 t + \sin^2 t + \sin t + 0$$

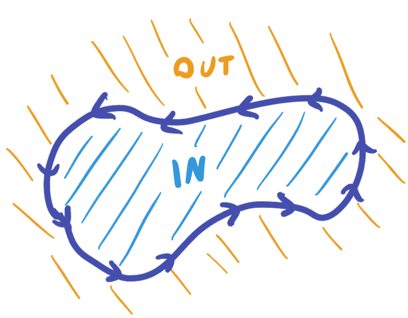
And integrating this over the curve yields the circulation

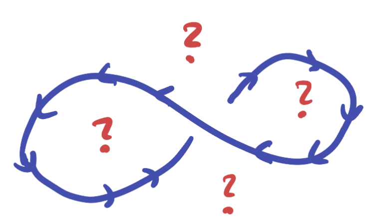
$$\int_0^{2\pi} -\cos t \sin^2 t + \sin^2 t + \sin t \, dt = -0 + \pi + 0 = \pi$$

(While it won't be relevant to our course, this definition *continues to make sense in all dimensions*, via the exact same formula).

Turning to flux, it perhaps comes as no surprise that things must change. There are both conceptual and formulaic problems with trying to naively apply our earlier formulation:

- (1) The formula involves a cross product, and this changes from a scalar quantity to a vector quantity in 3 dimensions, but we still want our overall integral to be a scalar.
- (2) Flux is supposed to measure the flow *through* something. But a curve does not separate two regions in 3D space like it does in the plane, so there's no natural inside and outside.



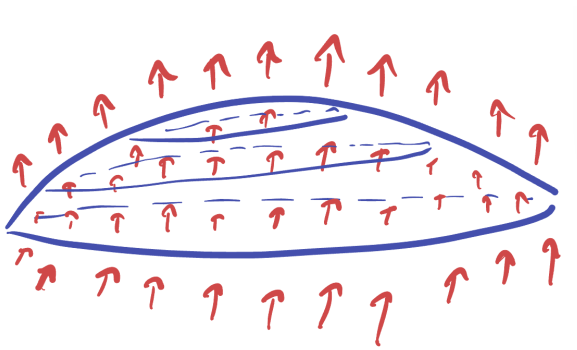


Both of these problems are fixed by realizing that the natural notion of a flux integral in three dimensions needs to be a *surface integral* instead of a *line integral.* This is the only change to the definition: flux is still the normal component of the vector field

Definition 19.6 (Flux Integrals in 3D)**.** The flux of a vector field $\vec{F}$ through a surface S is given by

$$\iint_S \mathrm{comp}_N(F)\, dS$$

where N is a normal vector to the surface, and dS the infinitesimal surface element.



Using our previous experience, we should try to simplify this formula a bit, hoping that any square roots hidden in the definition of the normal component cancel out with square roots in the definition of dS. To be explicit about this, we'll write this down for the case that S is the graph of a function $z = g(x, y)$ like we considered previously. A short review here is helpful: we found the surface area element dS by first finding two *tangent vectors* to the surface at a point

$$T_x = \langle 1, 0, g_x \rangle \qquad T_y = \langle 0, 1, g_y \rangle$$

and then finding the area of the parallelogram they span by taking the magnitude of their cross product

$$\text{Area} = |T_x \times T_y| = \sqrt{1 + g_x^2 + g_y^2}$$

For us now, its very convenient to note that before taking this magnitude, the cross product is itself a *normal vector to our surface* (in fact, it is the *upward facing normal*)

$$N = T_x \times T_y = \langle -g_x, g_y, 1 \rangle$$

If we use *this* normal vector, the simplifications that happen in our formula become more apparent:

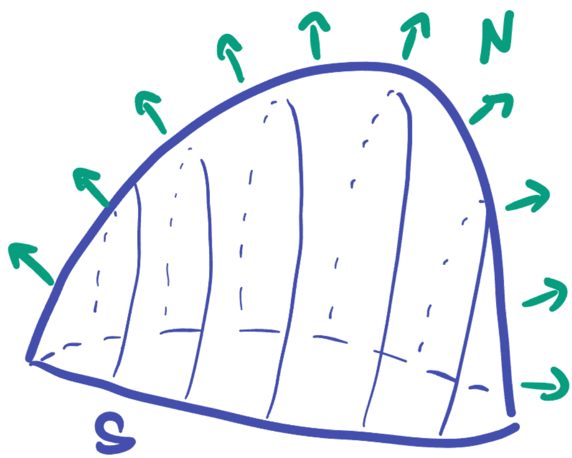
$$\mathrm{comp}_N(F) dS = \frac{F \cdot N}{|N|} dS = \frac{F \cdot N}{|N|} |N| dx dy = F \cdot N dx dy$$

This gives an easy formula to calculate the flux integral over a surface $z = g(x, y)$, very analogous to the case one dimension down

Theorem 19.3 (Flux Integrals in 3D)**.** *The flux of the vector field F through a surface S of a graph $z = g(x, y)$ can be computed as*

$$\iint_S \vec{F} \cdot N dxdy$$

Where $N = \langle -g_x, g_y, 1\rangle$ is the upward facing normal.



The upward facing normal.

At this point an example is in order

Example 19.4. Calculate the flux of $\vec{F} = \langle x, z, 1\rangle$ through the surface $z = 1 - x^2 + y^2$ for $z \geq 0$, with an upward pointing normal vector.

We can compute the vector field on our surface by substituting $z = 1 - x^2 - y^2$ into $\vec{F}$:

$$F(x, y, z) = F(x, y, 1 - x^2 - y^2) = \langle x, 1 - x^2 - y^2, 1\rangle$$

Computing the normal vector, $z_x = -2x$, $z_y = -2y$ and

$$N = \langle 2x, 2y, 1\rangle$$

This gives the integrand

$$F \cdot N = \langle x, 1 - x^2 - y^2, 1\rangle \cdot \langle 2x, 2y, 1\rangle = 2x^2 + 2y(1 - x^2 - y^2) + 1$$
$$= 2x^2 - 2x^2y - 2y^3 + 2y + 1$$

The flux integral is then just this over the domain, which is the unit circle R in the xy plane (what we get from setting $z = 1 - x^2 - y^2$ and $z = 0$ equal to one another).

$$\iint_R 2x^2 - 2x^2y - 2y^3 + 2y + 1dxdy$$

This integral is best computed by converting to polar coordinates.

This same idea works for general parametric surfaces with little change: if we write S using a parameterization

$$r(u, v) = (x(u, v), y(u, v), z(u, v))$$

we can find tangent vectors T_u and T_v just as before

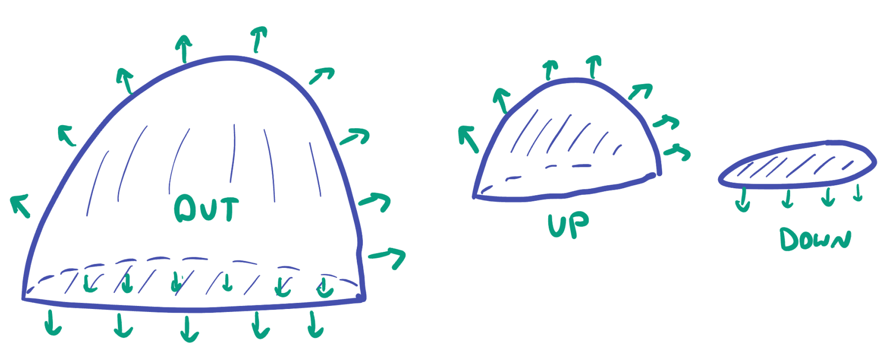
$$T_u = \partial_u r = \langle x_u, y_u, z_u \rangle \qquad T_v = \partial_v r = \langle x_v, y_v, z_v \rangle$$

and define the normal vector $N = T_u \times T_v$.

Theorem 19.4 (Flux Integrals over Parametric Surfaces)**.** *The flux of the vector field F through a surface S parameterized by $r = (x(u, v), y(u, v), z(u, v))$ is*

$$\iint_S \vec{F} \cdot N dudv = \iint_S \vec{F} \cdot (T_u \times T_v)\, dudv$$

A final note: the most common use case (in physics, and electrical engineering) involves S being a *closed surface*, where like for closed curves, we will take the default to mean *outward facing normal*. When our surface is made of multiple pieces, we'll need to make sure that we give each piece the correct normal vector (sometimes this will have to be the *downwards facing normal vector*)



Making sure to get the outward normal everywhere means drawing a picture and figuring out which pieces need *upwards* vs *downwards* normals.

Example 19.5. Calculate the flux of $\vec{F} = \langle x, z, 1 \rangle$ through the closed surface bounded by $z = 1 - x^2 + y^2$ for $z \geq 0$ and the unit disk in the $z = 0$ plane, with outwards facing normal.

The top half of this problem is exactly the same integral as in the previous example. To do the bottom half, we deal with the unit disk D in the xy plane. To make sure the entire normal vector is pointed *outward* we actually need the downwards normal here (since the upper half of the surface used the upwards normal).

So, for this second integral $N = \langle 0, 0, -1 \rangle$ and $dS = dxdy$ (since we are in a piece of the xy plane). Computing the vector field on this region is as simple as substituting $z = 0$, giving $F = \langle x, 0, 1 \rangle$. Taking the dot product gives

$$F \cdot N = \langle x, 0, 1 \rangle \cdot \langle 0, 0, -1 \rangle = -1$$

and so the surface integral for the bottom half of the surface becomes

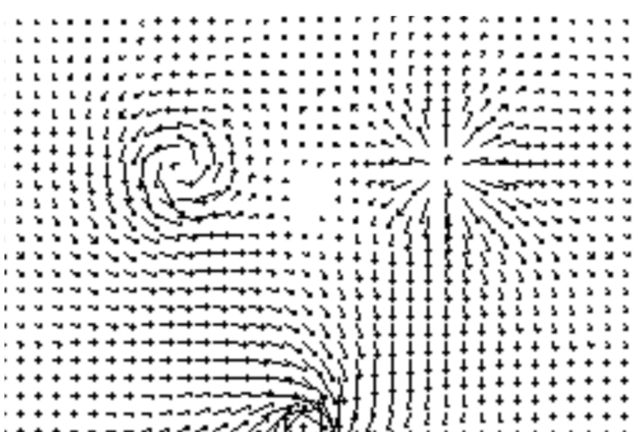
$$\iint_R (-1) dxdy$$

This is just -1 times the area of the unit disk, or $-\pi$.

20. Divergence and Curl

Vector fields are complicated objects as they can vary in both *magnitude* and *direction* at each point. We wish now to use calculus to help us get a better understanding of them: specifically by using derivatives to help us understand how a vector field can affect objects it pushes on.

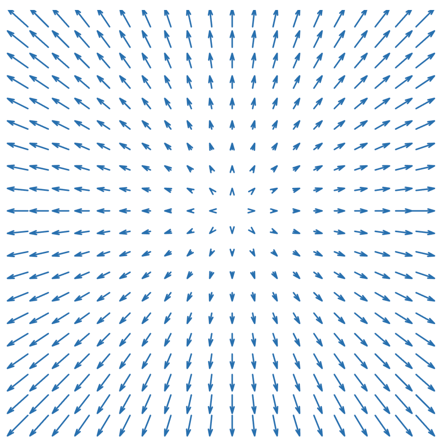
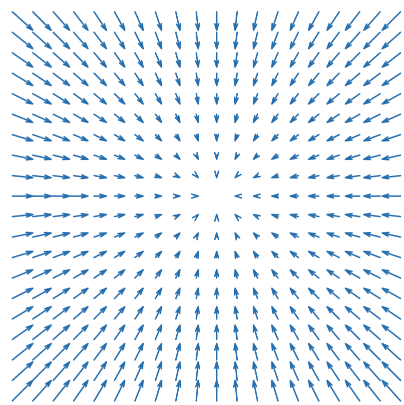
When we look at a vector field we can qualitatively distinguish different types of behavior going on: for example, in the field below there are regions where it looks like the field is swirling, as well as regions that look like the field is flowing at a constant speed / direction, and yet other regions where it looks to be *expanding*, or flowing away from a point.



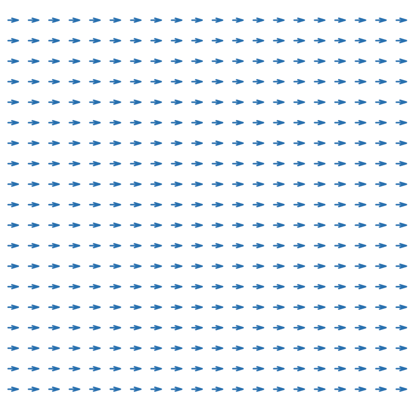
It's helpful sometimes to think about the behavior of a vector field by imagining it representing a fluid flow: then we can try to quantify these different ways the fluid can be changing: is it *spreading out*, or is it *spinning*. We formalize these below using partial derivatives into the concept of *divergence* and *curl*.

20.1. Divergence

The *divergence* of a vector field is a scalar quantity which measures how much the vector field is spreading out or contracting at a point. Imagine a small cloud of particles being blown around by the vector field. If that cloud *expands* over time, the divergence of the vector field is *positive* where the cloud is. If its volume *contracts*, the divergence is negative, and if it stays the same volume the divergence is *zero*. Here are some motivating examples

Vector fields with positive divergence (left) and negative divergence (right).



A vector field with zero divergence.

How do we come up with an equation that could measure this, for a vector field $\vec{F} = \langle P, Q \rangle$? One idea is to look at a small square around a point, and try to understand how much fluid is flowing in or out of that square. If theres a *net outflow* the fluid is expanding, a *net inflow* means its contracting, and *same flow in as flow out* means there's no divergence. To compute this, we compute the net flow through the horizontal faces as the *difference of the vectors there* and the net flow through the vertical faces as the *difference of the vectors found there.*

PICTURE

The total flow is then the sum of these, and in the limit as the size of the box decreases to zero, these differences become *derivatives*, giving

$$\langle P, Q\rangle \mapsto \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}$$

This quantity is the *divergence*, and can be given an easy-to-remember notation in terms of $\nabla = \langle \partial_x, \partial_y\rangle$. We are taking the x derivative of the first component and the y derivative of the second and then adding the results: this looks an awful lot like a dot product!

Definition 20.1 (Divergence)**.** The divergence of a vector field $\vec{F} = \langle P, Q\rangle$ is

$$\nabla \cdot \vec{F} = \langle \partial_x, \partial_y\rangle \cdot \langle P, Q\rangle = \partial_x P + \partial_y Q$$

Computing the divergence is no more difficult than computing partial derivatives

Example 20.1. Compute the divergence of the following vector fields:

- $\langle x, y\rangle$
- $\langle y, x\rangle$
- $\langle x^2, xy\rangle$
- $\langle x^2 + 3xy - 2, 4x^3 - 6xy^3 - 4y\rangle$

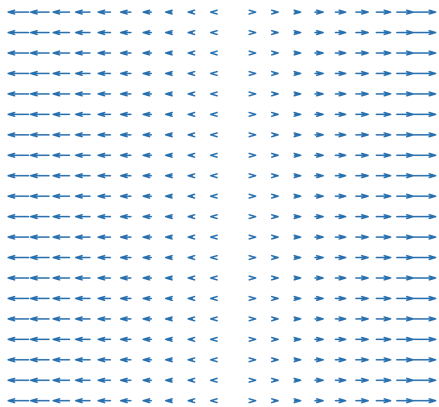
Now with a formula in hand, we can make sure we understand cases that might not match our initial intuition. Importantly, for a vector field to have divergence it doesn't have to *look like its spreading out*: it just has to have more net outflow than inflow.

Finally its of note that the definition of divergence generalizes directly to higher dimensions, so for 3D we have

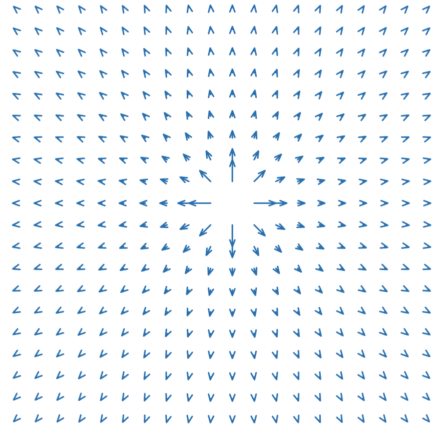
$$\nabla \cdot \vec{F} = \partial_x P + \partial_y Q + \partial_z R$$

and in n dimensions, if $\vec{F} = \langle F_1, F_2, \dots, F_n\rangle$ then

$$\nabla \cdot F = \sum_{i=1}^{n} \partial_{x_i} F_i$$



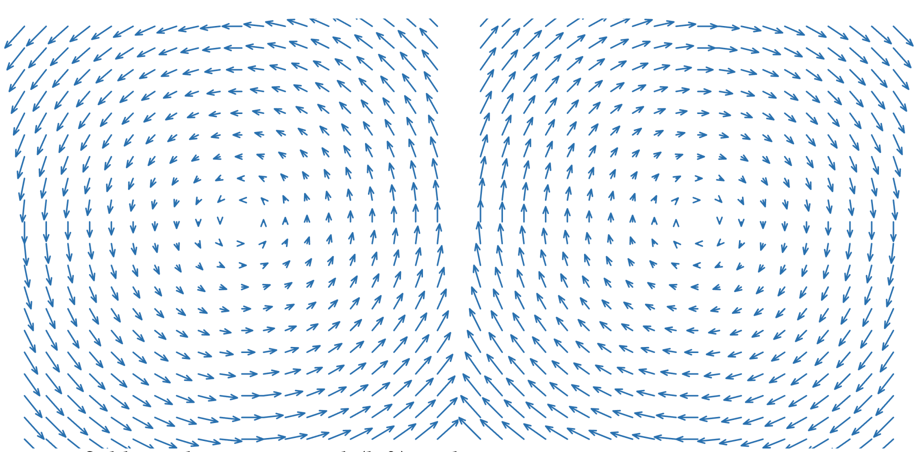
The vector field on the left has *positive divergence*: more fluid leaves each small box than enters.



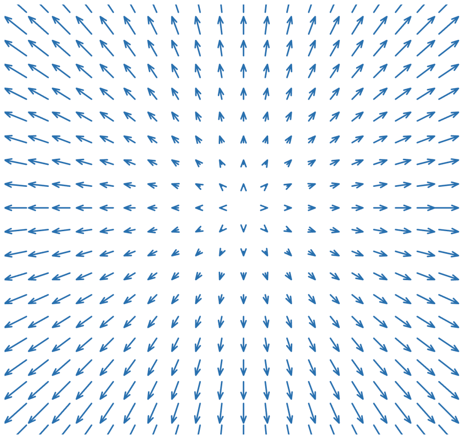
The vector field on the right has *zero divergence*: even though the vectors are spreading out they are getting shorter, so the net flow through each small box is zero.

20.2. Curl

Let's return to imagining little balls flowing around in a vector field again. We've found a way to quantify if they spread out *from eachother* or not, and our next goal is to capture their rotational motion. Here its helpful to think just of a single ball at a time flowing along with the fluid flow. For a 2 dimensional vector field we can imagine three possible cases: the flow could cause the ball to start to spin counterclockwise, spin clockwise, or glide along the flow without spinning. We will call these *positive curl*, *negative curl* and *zero curl* respectively.

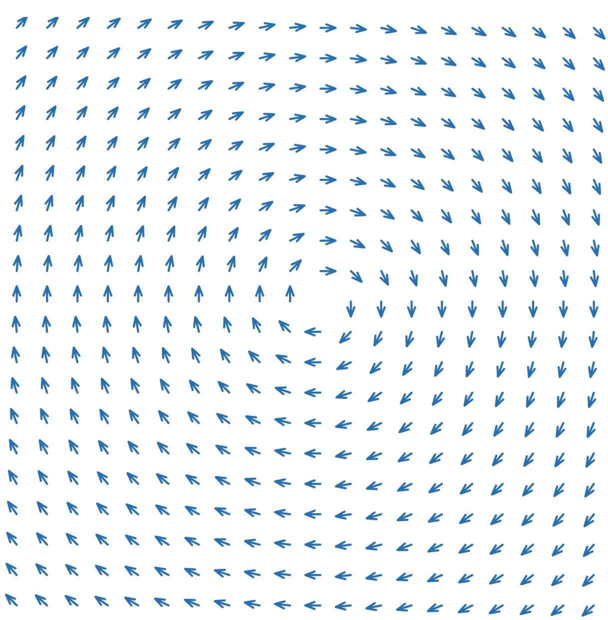


Vector fields with positive curl (left) and negative curl (right)

 A vector field with zero curl.

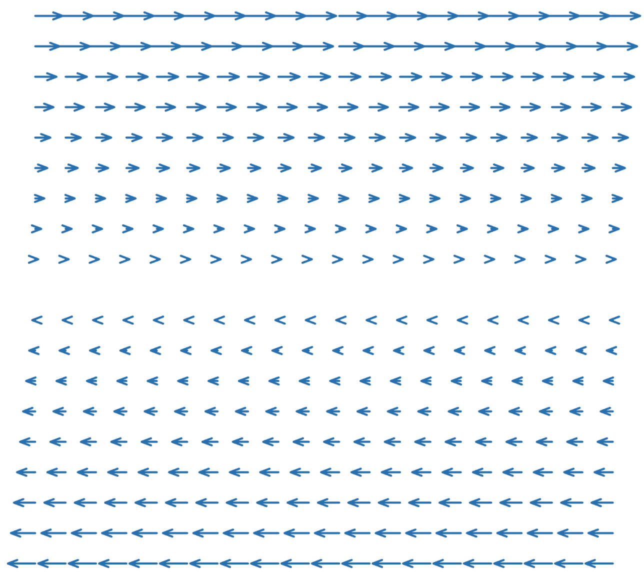
While seeing a vector field 'look like its spinning' can sometimes be a good indicator of curl, its important to remember this isnt exactly what we are looking for. Curl is about the *local properties* of a vector field: if it causes a small object to spin, not if its spinning itself. Indeed, consider the following vector field

$$\vec{F} = \frac{y}{\sqrt{x^2 + y^2}}\hat{\imath} + \frac{-x}{\sqrt{x^2 + y^2}}\hat{\jmath}$$



this vector field consists solely of *unit vectors*, and if you drop a ball in the field the ball will travel in a circle, but it *will not rotate about its own axis* as the vectors pushing it on each side are contributing equally. That is, even though the vectors are going around in a circle, it has *zero curl*.

A vector field on the other hand that *does cause something to spin* even though it does not go in a circle is $\vec{F} = \langle y, 0\rangle$. Here a ball dropped in the fluid will move along a horizontal line, but the vector pusing on its top will be a different length than the vector pusing on its bottom, which induces rotation. (For me its helpful to think of nearby vectors as being two friends pushing on each of my shoulders: if they push in the same direction with the same strength I won't spin, but if one pushes stronger than the other I will!)



With a clear qualitative picture in mind, we need to get quantitative about this. What sort of derivative measures rotational motion? Here, we find a use for our old friend the cross product

Definition 20.2 (2 Dimensional Curl)**.** If $\vec{F} = P\hat{\imath} + Q\hat{\jmath}$ is a 2 dimensional vector field, its *curl* is the scalar function

$$\nabla \times \vec{F} = \begin{vmatrix} \partial_x & \partial_y \\ P & Q \end{vmatrix} = Q_x - P_y$$

Example 20.2. Compute the divergence of the following vector fields:

- $\langle x, y\rangle$
- $\langle y, x\rangle$
- $\langle x^2, xy\rangle$
- $\langle x^2 + 3xy - 2, 4x^3 - 6xy^3 - 4y\rangle$

This definition also generalizes to 3 dimensions, but not as easily as divergence. Recall that the cross product behaved quite differently between two and three dimensions: in 2d it was simply a number (the signed area of the paralleogram spanned by the input vectors) whereas in 3 dimensions it was a vector (whose length was that area, and whose direction was *orthogonal* to the two inputs). Likewise, the three dimensional curl is no longer a scalar but a vector: its magnitude gives the *rate* of rotation and its direction gives the *axis*.

Definition 20.3 (3 Dimensional Curl)**.** If $\vec{F} = \langle P, Q, R\rangle$ is a 3-dimensional vector field, its curl is given by the vector field

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{vmatrix}$$

This definition actually fits together nicely with our 2D definition: given a two dimensional vector field $\vec{F} = \langle P, Q\rangle$, you can place it in the xy plane in three dimensions by constructing the vector field $\langle P(x,y), Q(x,y), 0\rangle$. As is familiar from our study of equations, the lack of z in this vector field means that the result is *the vectors are horizontal, and the same on every plane parallel to the xy plane.* Taking the curl of this yields

$$\nabla \times \langle P, Q, 0\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & 0 \end{vmatrix} = (Q_x - P_y)\hat{k} = (\nabla \times \vec{F})\hat{k}$$

That is, repeating a 2D vector field vertically gives a vector field whose curl points in the z direction (this is the axis of rotation for a horizontal vector field, so that makes sense!) and its magnitude is simply our familiar 2-dimensional curl $Q_x - P_y$.

Beyond three dimensions, curl becomes a much more complicated object to describe: already in four dimensional space the curl turns out to be a six dimensional vector! To be able to understand the curl in dimensions greater than three requires the more sophisticated langauge of *differential forms*, and is beyond the scope of our course. However, luckily most interesting applications of vector fields to the everyday world around us occur only in 2 and 3 dimensions, and so we will find plenty to discuss staying within this restricted realm.

20.3. Potentials and Antiderivatives

We now have three types of derivative that relate scalar and vector fields:

- **The Gradient** ∇f, which takes a scalar field and outputs a vector field.

- **The Divergence** $\nabla \cdot \vec{F}$ which takes a vector field and outputs a scalar field.
- **The Curl** Which takes in a vector field and outputs a scalar (2D) or a vector (3D).

We are on our way to study *integrals* of vector fields, so it probably comes as no surprise that we might be interested in *antiderivatives.* Recall from calculus 1 that a function g is an *antiderivative* of f if $\frac{d}{dx}g = f$. Here we will take a little time to contemplate finding antiderivatives for *gradient, divergence, and curl.* First, a bit of terminology: following physics we traditionally do not call these *antiderivatives* but rather *potentials* (as they arise in describing *potential energy*, and a *potential for the electromagnetic field*).

Note to anyone feeling a bit overwhelmed with everything towards the end of a semester: the only one of these we are *actually going to need to do often* is to find an antiderivative for the gradient; so if you understand the first case well you should be fine.

20.3.1. The Gradient & Conservative Vector Fields

Let's start with the gradient:

Definition 20.4 (Potential (for Gradient))**.** A function f is a potential, or antiderivative, for the vector field $\vec{F}$ if $\nabla f = \vec{F}$.

For example, since we know $\nabla(x^2 + y^2) = \langle 2x, 2y\rangle$, we can say that the vector field $\vec{F} = \langle 2x, 2y\rangle$ comes from the *potential function* $f = x^2 + y^2$. Just like derivatives of ordinary functions, potentials are not unique: the function $x^2 + y^2 + 5$ is also a potential for the same vector field $\vec{F}$.

How can we find potentials more systematically? For a vector field $F = \langle P, Q\rangle$ to be the gradient $\nabla f = \langle f_x, f_y\rangle$ of some potential f, we would need $P = f_x$ and $Q = f_y$. These do not directly specify f but rather specify its x and\$ y partial derivatives: this suggests we can attempt to solve for f via integration. Its easiest to see in an example

Example 20.3. Find a potential for the vector field $\langle 2xy + y, x^2 + 1\rangle$

If there is such an $f(x, y)$ then $f_x = 2xy$ and $f_y = x^2$. Integrating,

$$f = \int f_x dx = \int 2xy\, dx = x^2y + C$$

$$f = \int f_y dy = \int x^2y\, dy = x^2y + K$$

Thus, in the one case we have found $f = x^2y$ (perhaps plus a constant) and in the other we *also* see $f = x^2y$ (perhaps plus a constant). Thus we have it $f = x^2y$ is a potential for F.

Sometimes we have to be a bit more careful with the constants:

Example 20.4. Find a potential for the vector field $\vec{F} = \langle 2xy + 1, x^2 + 4y^3 \rangle$

Applying the same trick we integrate f_x and f_y to find

$$f = \int 2xy + 1\, dx = x^2y + x + C$$

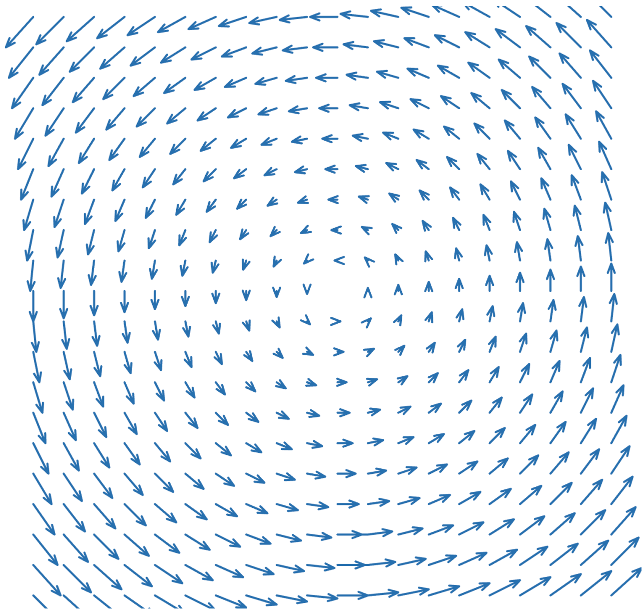
$$f = \int x^2 + 4y^3\, dy = x^2y + y^4 + K$$

Here at first our two expressions for f seem not to agree. But thinking a bit harder, we recall that C is a constant of integration from a dx integral, so C can actually depend on y, and similarly K is a constant of integration from a dy integral so it could have x's in it! With this realization, we see the solutions actually are consistent with $C = y^4$ and $K = x$, giving the potential

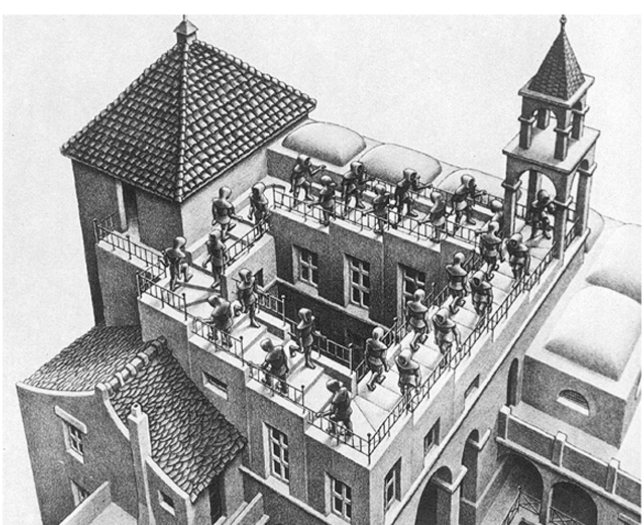
$$f = x^2y + x + y^4$$

So far, finding potentials seems to fit well within our general framework of *to do something in multivariable calculus, you just have to do a Calculus I problem multiple times*: instead of just finding one integral we now have x and y derivatives, so we need to take both an x and y integral and compare the results. But there is an important *new* subtlety here in the multivariable case: not all vector fields have an antiderivative! (Compare this to single variable calculus, where an antiderivative always exists, even if you struggle to write it down).

The reason for this is actually pretty intuitive, so long as we recall the *geometrical meaning* of the gradient, which points in the direction of steepest increase of f. Consider the vector field below, where all the vectors turn around in a big circle.



If there were a potential f for this vector field, then these arrows would be pointing in the direction of steepest increase. So, following them around a circle you should expect to find the value of f getting bigger and bigger (imagine hiking on the graph of f: you're going uphill the whole time). This leads to trouble after we've done a full circle! At that point we should be back to our original location (and so at your original elevation) yet you got there by walking continuously uphill! Such things may exist in art (such as this sketch of an impossible staircase by Escher) but cannot exist in reality. Thus, our circular vector field cannot have come from a potential.



Let's try this example mathematically, and see where things go wrong. A circular vector field is $\vec{F} = \langle y, -x \rangle$, and if we were to imagine that $\vec{F} = \nabla f$ this would imply $f_x = y$ and $f_y = -x$. Trying to solve for f by partial integration gives

$$f = \int y dx = xy + C(y) \qquad f = \int -x dy = -xy + C(x)$$

Thus f has to equal xy (possibly plus some stuff only involving a y) and *simultaneously* has to equal $-xy$ (possibly plus some stuff only containing x). There is no such function since $xy \neq -xy$, so f cannot exist. This confirms that in contrast to Calc I, having an antiderivative in multivariable calculus is a *special property* and is not guaranteed. We call such special fields *conservative*.

Definition 20.5 (Conservative Vector Fields)**.** A vector field $\vec{F}$ is conservative if it is the gradient of some function f, that is if $\vec{F}$ has a potential, or

$$\vec{F} = \nabla f$$

It would be nice to have a means of determining *when* a vector field has an antiderivative, and when it does not. Luckily there is an easy calculation to do so.

Theorem 20.1 (Curl of the Gradient is Zero)**.** *If $f(x, y)$ is a function of two variables, then*

$$\nabla \times \nabla f = 0$$

We check this with a quick calculation: $\nabla f = \langle f_x, f_y \rangle$ and then

$$\nabla \times \nabla f = \nabla \times \langle f_x, f_y \rangle = \begin{vmatrix} \partial_x & \partial_y \\ f_x & f_y \end{vmatrix}$$

$$= \partial_x f_y - \partial_y f_x = f_{yx} - f_{xy} = 0$$

Where the final equality is true because the order of partial derivatives does not matter!

Exercise 20.1. Check this still holds true in three dimensions, for $f = f(x, y, z)$.

$$\nabla \times \nabla f = \langle 0, 0, 0 \rangle$$

This is useful to us because it gives us a definite check for when a potential cannot possibly exist: if $\nabla \times F \neq 0$ then there is no chance that $F = \nabla f$ for some f, as the curl would have to be zero! The converse of this also holds, so long as the vector field is defined everywhere

Theorem 20.2 (Existence of a Potential)**.** *Let $\vec{F}$ be a vector field defined (and differentiable) everywhere on $\mathbb{R}^2$ or $\mathbb{R}^3$. Then it is possible to find a potential for $\vec{F}$ if and only if $\nabla \times \vec{F} = 0$*

One must be careful in applying this theorem however: its crucial that the vector field actually be defined everywhere: check for yourself that the vector field of rotating unit vectors below has zero curl, even though we can see it is not a gradient (as it goes in a circle!) This does not contradict our theorem because this vector field is *not defined everywhere*: it has a division - by - zero problem at the origin!

$$\vec{F} = \left\langle \frac{y}{\sqrt{x^2+y^2}}, \frac{-x}{\sqrt{x^2+y^2}} \right\rangle$$

20.3.2. OPTIONAL: Undoing Divergence

Lets turn to investigate a similar question for the divergence: this type of derivative takes a vector field to a scalar field, so the question we should be asking is *given a scalar field f, does there exist a vector field $\vec{F}$ where $\nabla \cdot F = f$? Such a vector field would be an* antiderivative with respect to divergence. *This does not seem to have a standard name, but one could also call it a* divergence potential*.

Definition 20.6 (Potential for Divergence)**.** Given a scalar field f, a *divergence potential* is a vector field F such that $\nabla \cdot F = f$.

Again we begin with a simple example: say $f(x, y) = x + \sin(y)$: can we produce a potential $\vec{F}$ for this with respect to divergence? If there were such a $\vec{F} = \langle P, Q \rangle$, this would imply $\nabla \cdot \vec{F} = P_x + Q_y = x + \sin(y)$, so we have *one* equation to solve for *two unknowns.* Such a thing is usually easy to solve: for instance here we could say $P_x = x$ and $Q_y = \sin(y)$ to get $P = x^2/2$ and $Q = -\cos(y)$ to get $F = \langle x^2/2, -\cos y \rangle$. Or we could take $P_x = x + \sin(y)$ and $Q_y = 0$ yielding another solution $F = \langle x^2/2 + x\sin(y), 0 \rangle$; such vector fields are highly non-unique.

Indeed this second trick shows one way we can *always* find a vector field whose divergence is f - no matter what f is. If we set $P = \int f(x, y)dx$ and $Q = 0$ then

$$\nabla \cdot \langle P, Q \rangle = \frac{\partial}{\partial x} \int f(x, y)dx + \frac{\partial}{\partial y} 0 = f + 0 = f$$

In stark contrast to the case of the gradient, its very easy to find an antiderivaive for divergence! Its of note that the non-uniqueness here is pretty interesting; we found two vector fields whose divergence is $x + \sin(y)$ above, but the two vector fields behaved *very differently*: the ambiguity in antiderivative isn't just in a single $+C$ anymore! Investigating this further would take us too far afield so we will not, but for anyone interested, this is just the start of a deep mathematical theory involving *real analysis* and *topology*, that has proven very useful in fundamental physics.

20.3.3. OPTIONAL: The Curl and Vector Potentials

Last but certainly not least, we can consider the same type of question for curl. This one does have a standard name due to its use in physics, and is simply called the *Vector Potential* (though note this term could have equally well applied to the divergence, so one may wish to call it the *Curl Potential* or *Vector Potential for Curl* to avoid confusion).

Definition 20.7 (Vector Potential (For Curl))**.** In two dimensions, given a scalar field g, a vector field $\vec{F}$ is a *vector potential* for g if $\nabla \times F = g$, so F is an antiderivative of g for the curl derivative.

In three dimensions, since curl returns a vector field, we have to consider antiderivaties of *vector fields* instead: given a vector field G, a vector field F is a vector potential if $\nabla \times F = G$.

The existence of a vector potential in this case depends on the dimension. For 2D vector fields (where the curl is a scalar), one can always find a vector potential using the same trick we did for the divergence.

Example 20.5. Find a vector potential F for the scalar field x^2y.

A potential would be a vector field $F = \langle P, Q \rangle$ where $\nabla \times F = Q_x - P_y = x^2y$. We just have one equation to satisfy here so its easy to make up a solution: one is just to set $Q_x = x^2y$ and $P_y = 0$. Then $Q = \int x^2 y dx = x^3y/3$ and $P = 0$ giving

$$F = \langle x^3y/3, 0 \rangle$$

In 3D, we have to contend with curl being a vector, which gives a system of three equations that need to simultaneously be solved:

Exercise 20.2. Find a vector potential F for the vector field $\vec{G} = \langle y, z, x \rangle$.

Its easy to imagine that this may no longer always be possible, and indeed its not: a computation with divergence and curl provides an obstruction:

Theorem 20.3. *Let $\vec{F}$ be a 3D vector field. Then* $\nabla \cdot \nabla \times F = 0$

Exercise 20.3. Check this, for an arbitrary vector field $\vec{F} = \langle P, Q, R \rangle$.

We can use this just like the identity for curl and the gradient, to give a strict constraint on when a vector field G cannot have a vector potential.

Theorem 20.4. *Let G be a 3D vector field. Then if $\nabla \cdot G \neq 0$, it is impossible to find a vector potential $G = \nabla \times F$ for G.*

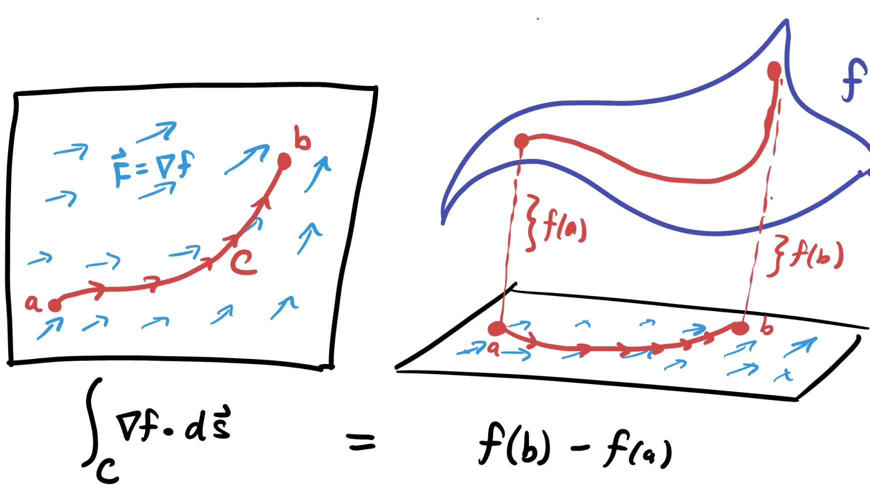
21. Fundamental Theorems

We've already made great progress in the theory of integrating vector valued functions by converting circulation/work integrals into

21.1. The Fundamental Theorem of the Gradient

Theorem 21.1 (Fundamental Theorem of the Gradient)**.** *Let f be a scalar function and $\vec{F} = \nabla f$ its gradient. Then if C is a curve from p to q, the line integral of F can be evaluated as*

$$\int_C \nabla f \cdot d\vec{s} = f(q) - f(p)$$



This is a direct analog of the 1 dimensional fundamental theorem of calculus which tells us we can evaluate an integral by finding an antiderivative and evaluating at the endpoints - we've just generalized to replace the interval $[a, b]$ along the real line with a curve C, and the derivative of a function with the gradient.

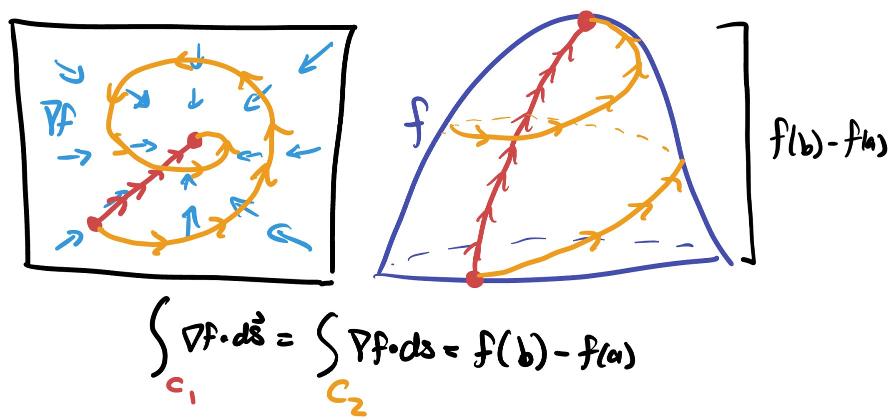
In fact, the *proof* of the original fundamental theorem generalizes directly to this case as well, as we saw in class: $\nabla f \cdot c'$ is the directional derivative of f in the direction of the curve c, and integrating this over its length adds up all these infinitesimal changes in height to get the *net* change in height: the value of f at the endpoint minus the value of f at the start.

There is one subtlety in higher dimensions which did not occur in the one dimensional case. For 1 variable functions every continuous function has an antiderivative, so we can always apply the fundamental theorem if we can find it. But for vector fields we've already seen that *not every vector field has a potential*: indeed, a necessary condition for a continuous vector field to have a potential is that its curl is zero. (This is a useful concept, so its nice to have a single word for it: recall we call such vector fields *conservative*).

Definition 21.1. A vector field F is called *conservative* if it has an antiderivative with respect to the gradient. Any such antiderivative f with $\nabla f = F$ is called a *potential* for F. If F is continuous and defined everywhere, then F is conservative if and only if its curl is zero.

The fundamental theorem for the gradient makes it easy to evaluate integrals if you can find the potential. But it also has some interesting theoretical consequences:

Corollary 21.1 (Path Independence)**.** *If $\vec{F}$ is a conservative vector field then the value of its line integral along a curve C between points p and q is* independent of the path.



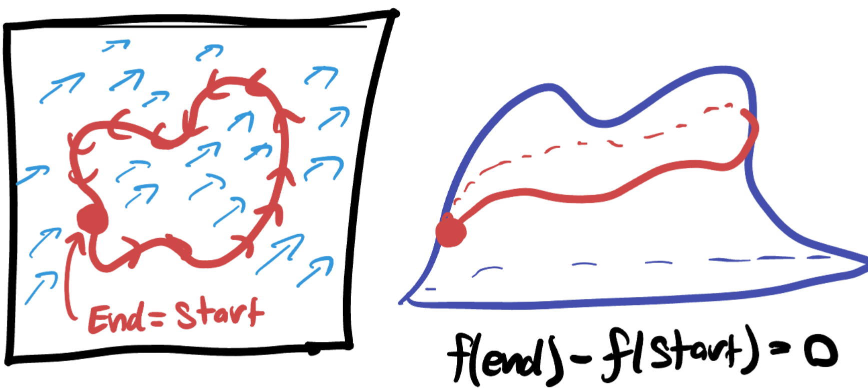
This is a rather striking property at first - it means for these vector fields it doesn't even matter what path we integrate them along: so long as the paths start and end at the same point the circulation integral will have the same value!

Example 21.1. Compute the line integral of $(3 + 4xy^2)i + 4x^2yj$ along the arc of the unit circle joining $(1, 0)$ to $(0, 1)$.

Example 21.2. Integrate the circulation of $F = \langle x + 1, y^2 - 2y - 1\rangle$ along the curve $c(t) = \langle t^2 - 2t, t + 1\rangle$ for $t \in [0, 2]$.

Corollary 21.2 (Integrating around Closed Curves). *Let C be a closed curve and $\vec{F}$ be a conservative vector field. Then*

$$\oint_C \vec{F} \cdot d\vec{s} = 0$$



Like above, the proof is simple: since $\vec{F}$ is conservative there is a potential $\nabla f = \vec{F}$ and so we can apply the fundamental theorem. But since C is a closed curve, its endpoint equals its start point, so $f(\text{end}) = f(\text{start})$ and their difference is zero. This is quite useful as it lets us often evaluate difficult looking integrals with almost no work at all.

Example 21.3. Compute the integral of $\vec{F} = \langle x^4y^5, x^5y^4 \rangle$ \$ around the closed curve $c(t) = \langle 1 + \cos(2t) + \sin(t), \sin^2(t) + \cos(t) - 1 \rangle$ for $t \in [0, 2\pi]$.

Solution: Since the curve is closed, we check if the vector field is conservative by computing its curl:

$$\nabla \times \vec{F} == 0$$

Since F is defined everywhere and $\nabla \times F = 0$, we know that a potential exists, and so the integral around any closed loop is zero: we're done!

Of course this doesn't work if the curl turns out to be nonzero, as then the vector field is *not* conservative, and there exists no potential for which to apply the fundamental theorem. In such cases we can always go back and just evaluate the line integral with our original method, finding $F(c(t)) \cdot c'(t)$ and integrating along the curve. But there's also *another* fundamental theorem that can come to our aid, discussed below.

21.2. The Fundamental Theorem of the Curl

Recall that we called an antiderivative for the curl a *vector potential*: given a field F a vector potential is another field A such that $\nabla \times A = F$. This type of derivative comes with its own fundamental theorem as well, now relating *double integrals* to *single integrals*. To write it in language reminiscent of the original fundamental theorem, one might say

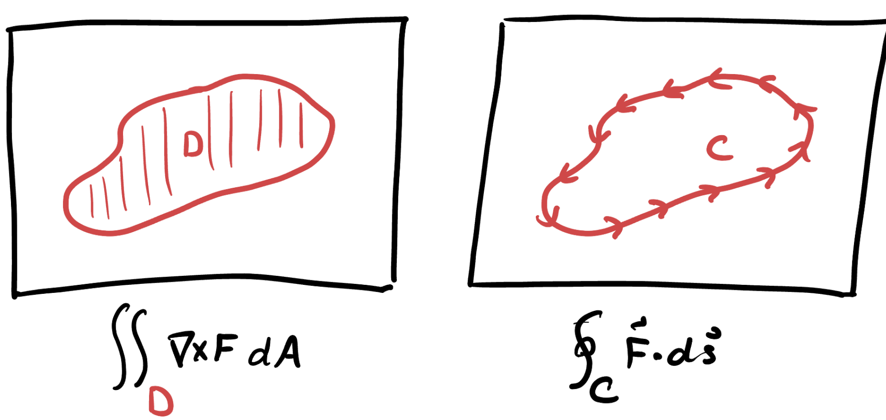
Let F be a field and A be a vector potential for F. Then if D is a region in the plane whose boundary is the closed curve C, one has

$$\iint_D F dS = \oint_C A \cdot ds$$

That is - taking hte *antiderivative* for curl removes one of the integrals! This theorem is more commonly stated without giving a separate name to the vector potential A, but instead writing out the curl explicitly:

Theorem 21.2 (2D Fundamental Theorem of the Curl (Green's)). *Let D be a region in the plane whose boundary is the closed curve C and F a vector field. Then*

$$\iint_D \nabla \times F dA = \oint_C F \cdot d\vec{s}$$



In two dimensions the curl is a *scalar*; indeed if $F = \langle P, Q \rangle$ one might write this as $\nabla \times F = Q_x - P_y$ giving the alternate form (same theorem, different symbols):

$$\iint_D (Q_x - P_y) dA = \oint_C P dx + Q dy$$

This theorem is often mostly used in reverse: compute a line integral *by turning it into a double integral!* The reason is, that double integral is just the integral of a *scalar function*, and we've known how to do those for quite some time!

Example 21.4. Compute the line integral $\oint_C xdy - ydx$ around the circle of radius 3 centered at the origin, parameterized counterclockwise.

Solution: We could evaluate this integral directly by parameterizing $c(t) = (3\cos t, 3\sin t)$. But the fundamental theorem says we could instead choose to *take the curl* and integrate over the interior of the circle c. Here $F = \langle -y, x\rangle$ and

$$\nabla \times F = \partial_x x + \partial_y y = 2$$

Thus our integral becomes $\iint_D 2dA$$ But integrating $2dA$ over a region just returns twice its area, and since our region is the circle of radius 2, this is

$$2(\pi(2^2)) = 8\pi$$

Example 21.5. Compute the circulation integral

$$\oint_C y^3 dx - x^3 dy$$

For C the circle of radius 2 about the origin, parameterized clockwise.

Solution: This line integral corresponds to the vector field $F = \langle y^3, -x^3\rangle$, whose curl is

$$\nabla \times F = \begin{vmatrix} \partial_x & \partial_y \\ y^3 & -x^3 \end{vmatrix} = 3y^2 + 3x^2 = 3(x^2 + y^2)$$

Using the fundamental theorem we can integrate this over the interior of the circle of radius 2 instead of integrating F along the boundary. This is easiest in polar coordinates where $3(x^2 + y^2) = 3r^2$ and

$$\iint_D \nabla \times F\, dA = \int_0^{2\pi} \int_0^2 3r^2 r dr d\theta = 24\pi$$

This is particularly useful when the line integral is around a piecewise curve: instead of doing multiple line integrals and adding them all up, you can take the curl, and integrate that over the interior.

Example 21.6. Compute the circulation of $F = \langle ye^x, 2e^x\rangle$ around the boundary of the unit rectangle with vertices $(0,0), (1,0), (1,1)$ and $(0,1)$ parameterized counterclockwise starting at $(0,0)$.

There is a direct analog of this theorem in three dimensions, unchanged except for the fact that the curl is a vector field

Theorem 21.3 (3D Fundamental Theorem of the Curl (Stokes's)). *Let C be a curve in 3D and S be a surface whose boundary is C. Then if F is a vector field*

$$\iint_S \nabla \times F \cdot d\vec{S} = \oint_C F \cdot d\vec{s}$$

This can be used to compute circulation integrals around a 3D curve, by using a (hopefully simpler) surface that spans them. But it also makes for really easy calculations of integrals over a *closed surface* (like a sphere, with no boundary).

Example 21.7. Let F be the vector field $F = \langle xy, y+1, z+x \rangle$ and compute the flux of $\nabla \times F$ through the sphere of radius 2 centered at the origin.

Solution: Stokes theorem says that the integral of $\nabla \times F$ over a surface S is the same as the circulation around the boundary. But a sphere has no boundary! So there is no circulation around this boundary, and the integral is zero. This means our original double integral is also zero!

These sorts of computations are very useful in physics and electrical engineering, when working the electromagnetic field. For example, when there is a current $\vec{J}$ flowing and the electric field is not changing in time, the magnetic field satisfies $\nabla \times B = J$, so the flux of *current* through a surface can be computed by integrating $\nabla \times B$ on that surface, and then stokes theorem sayst this is the same as the integral of B around the boundary. (We won't dive into such applications here, but for those of you continuing onto electromagnetism this spring, this is where that class will pick up.)

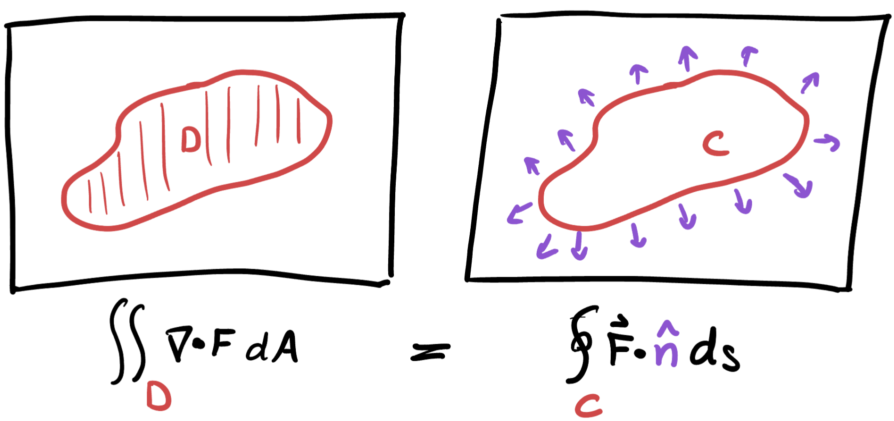
21.3. The Fundamental Theorem of the Divergence

We've seen so far a pattern in generalizing the fundamental theorem of calculus to vector fields: given a type of derivative, we search for a corresponding notion of *anti derivative*, and using this antiderivative allows us to *remove one integral*: the derivative and antiderivative "cancel". So perhaps it comes as no surprise that there is yet another fundamental theorem of calculus, this time paired with the divergence:

Theorem 21.4 (2D Fundamental Theorem of the Divergence)**.** *Let F be a vector field in the plane and $\nabla \cdot F$ be its divergence. Then given a closed curve C bounding a region D in the plane, the flux of F through C can be calculated by integrating its divergence over the interior D:*

$$\oint_C F \times d\vec{s} = \iint_D \nabla \cdot F \, dA$$

Recall that $\oint_C F \times d\vec{s}$ is shorthand for the flux integral, a dot product with the normal vector $\oint_C F \cdot n \, ds$.

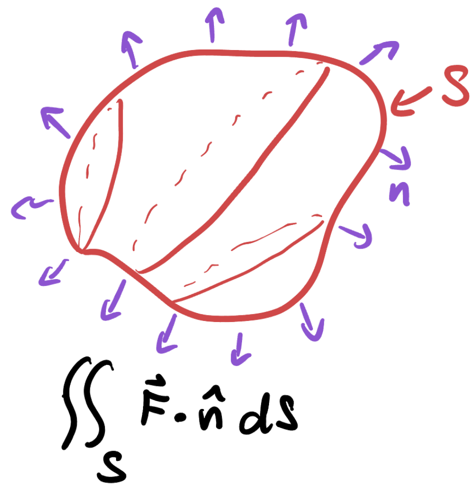


As written the theorem applies only to *positively oriented curves*: that is, those traversed counterclockwise. But its easy to use for negatively oriented curves if you remember that going the other way around the curve multiplies the line integral by -1.

This theorem also has a direct analog in three dimensions, replacing curve with surface and area with volume:

Theorem 21.5 (3D Fundamental Theorem of the Divergence (Gauss'))**.** *Let F be a 3D vector field and $\nabla \cdot F$ its divergence. Then given a closed surface S bounding a region E in space, the flux of F through S can be calculated by integrating its divergence over the interior E:*

$$\iint_S F \cdot dS = \iiint_E \nabla \cdot F \, dV$$



The intuition for this theorem is a little clearer than the case of curl, but essentially the same vibe: divergence measures how much a vector field spreads out at a point, and the flux integral captures how much it spreads out through a surface. The theorem just says that the net amount it spreads out through the curve/surface is the sum of the infinitesimal amounts it spreads out on the interior. This is again useful to take complicated line or surface integrals and replace them with easier double or triple integral of a scalar function.

Example 21.8. Use the divergence theorem to find the flux of $F = \langle x, y\rangle$ through the sides of the triangle with vertices $(0, 0)$, $(2, 0)$ and $(2, 1)$.

Example 21.9. Use the divergence theorem to calculuate the outward flux of $F = xye^z\, i + xy^2z^3\, j - ye^z\, k$ through the box defined by the coordinate planes and $x = 1$, $y = 2$, $z = 3$.

Often this triple integral can be easiest to compute by changing coordinates:

Example 21.10. Compute the flux of the vector field $F = \langle x, y, 1\rangle$ through the unit sphere.

Example 21.11. Compute the flux of $F = 3xy^2\, i + xe^z\, j + z^3\, k$ through the surface of the solid cylinder bounded by $x^2 + z^2 = 9$ and $x = 0$,$x = 3$.

21.4. The Bigger Picture

The original fundamental theorem of calculus in one variable emerged in the mid 1600s. Over the next two centuries the field matured and developed through various uses in multiple variables, and these new generalized versions of the fundamental theorem began to appear in rapid succession around the mid 1800s. While at the time each was discovered for its own bespoke use (often within electromagnetism) with our modern eyes we may take a look at the collection of fundamental theorems and see a hint of some deeper pattern:

$$\int_a^b f'dx = f(b) - f(a)$$

$$\int_C \nabla f \cdot d\vec{s} = f(b) - f(a)$$

$$\iint_D \nabla \times F\, dS = \oint_C F \cdot d\vec{s} \qquad \iint_S \nabla \times F \cdot d\vec{S} = \oint_C F \cdot d\vec{s}$$

$$\iint_D \nabla \cdot F dS = \oint_C F \cdot d\vec{s} \qquad \iiint_E \nabla \cdot F dV = \iint_S F \cdot d\vec{S}$$

On the left side of each of these we are integrating some derivative over a region (an interval, curve, surface, or volume). And on the right we are instead evaluating its *antiderivative along the boundary of the original region.* In the case of the interval or curve the boundary is just two points, so "evaluating on the boundary" is just subtracting their difference. On the higher dimensional regions the boundary is 1 or 2 dimensional, so "evaluating on the boundary" means doing a line or surface integral.

After noticing this pattern mathematicians went off in search of the correct framework to prove it in. One can already imagine several difficulties here in trying to unify the proofs of all these fundamental theorems: for one, curl behaves differently in dimension 2 (where its a scalar) and in dimension 3 (where its a vector). The correct language for combining discussions of vectors and scalars, as well as infinitesimal lengths, area elements and volume elements is called the theory of *differential forms* which developed over the mid-late 1800s. Differential forms are mathematical tools for working with

In 1889, Volterra discovered a fundamental theorem of calculus in this new language (often called the generalized Stokes Theorem)

Theorem 21.6 (Fundamental Theorem for Differential Forms)**.** *If ω is an $n-1$ differential form and M is an n dimensional region, then integrating $d\omega$ over M is the same as integrating ω over the (n-1 dimensional) boundary of M.*

$$\iint_M d\omega = \int_{\partial M} \omega$$

About a decade later Cartan realized in 1899 that the traditional language of vector calculus could all be rewritten within the theory of differential forms:

Theorem 21.7 (Differential Forms and Vector Calculus)**.**

- *Zero forms are functions*
- *1 forms are vector fields: and anything you can integrate over a curve.*
- *2 forms measure infinitesimal areas, and anything you can integrate over a surface*
- *3 forms measure infinitesimal volumes, and anything you can integrate over a 3D region*

The exterior derivative *of differential forms encodes familiar operations. In three dimensions:*

- *The derivative of a zero form is a 1 form: this corresponds to the gradient*
- *The derivative of a 1 form is a 2 form: this corresponds to the curl*
- *The derivative of a 2 form is a 3 form: this corresponds to the divergence*

A few decades later, in 1917, the mathematician Edouard Goursat realized that all of the previously known fundamental theorems of calculus were special cases of the fundamental theorem for differential forms, essentially showing that the entire theory of vector calculus truly belongs as a special case of this new more general technology.