

# **CALCULUS 3**

**VECTOR ANALYSIS**

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# Table of contents

<b>Preface</b>	<b>1</b>
<b>I. Space</b>	<b>3</b>
<b>1. Dimensions</b>	<b>7</b>
1.1. Higher Dimensions and Configurations . . . . .	8
1.2. Cartesian Coordinates . . . . .	9
1.3. Other Coordinate Systems . . . . .	11
1.4. Videos . . . . .	12
1.4.1. Cartesian Coordinates . . . . .	12
1.4.2. Higher Dimensions . . . . .	12
1.4.3. Distances . . . . .	12
<b>2. Vectors</b>	<b>13</b>
2.1. Arithmetic of Vectors . . . . .	14
2.2. Coordinate Bases . . . . .	15
2.3. Magnitude and Direction Information . . . . .	16
2.4. Videos . . . . .	17
2.4.1. Video Tutorial Series . . . . .	18
<b>3. Operations</b>	<b>19</b>
3.1. The Dot Product . . . . .	19
3.1.1. Measuring Angles . . . . .	20
3.1.2. Projections . . . . .	21
3.1.3. Standard Basis . . . . .	22
3.2. Cross Product . . . . .	22
3.2.1. Computing the Cross Product . . . . .	23
3.2.2. Geometry of the Cross Product . . . . .	25
3.2.3. The Standard Basis . . . . .	26
3.3. The Triple Product . . . . .	27
3.4. Videos . . . . .	28
3.4.1. Dot Products . . . . .	28
3.4.2. Cross Products . . . . .	28

*Table of contents*

3.4.3. Triple Products . . . . .	28
<b>4. Shapes</b>	<b>29</b>
4.1. Lines . . . . .	29
4.2. Planes . . . . .	30
4.3. Circles and Spheres . . . . .	31
4.4. Other Shapes . . . . .	33
4.4.1. Helpful Videos . . . . .	36
<b>II. Curves</b>	<b>37</b>
<b>5. Parameterization</b>	<b>39</b>
5.1. Parameterization Tips . . . . .	40
5.1.1. Same Curve, Different Parameterizations . . . . .	41
5.1.2. New Curves from Old . . . . .	42
5.2. Case Study: Spirals . . . . .	42
5.3. Videos . . . . .	44
<b>6. Calculus</b>	<b>45</b>
6.1. Limits . . . . .	45
6.2. Differentiation . . . . .	46
6.3. Integration . . . . .	48
6.4. Videos . . . . .	49
<b>7. Geometry</b>	<b>51</b>
7.1. Arc Length . . . . .	51
7.2. Curvature . . . . .	53
7.3. Framing a Curve . . . . .	54
7.4. Videos . . . . .	55
<b>8. Differential Equations</b>	<b>57</b>
8.1. Projectile Motion . . . . .	57
8.2. Universal Gravitation . . . . .	59
8.2.1. Deriving Newton's Gravitational Law . . . . .	59
8.2.2. The Length of a Year . . . . .	61
8.2.3. Living in 3-dimensions . . . . .	63
8.2.4. The 3-body Problem . . . . .	63
8.3. Electromagnetism . . . . .	64
8.3.1. Charges in an Electric Field . . . . .	64
8.3.2. Charges in a Magnetic Field . . . . .	65
8.3.3. The Discovery of Antimatter . . . . .	68

8.4. Weather . . . . .	68
<b>III. Multivariate Differentiation</b>	<b>71</b>
<b>9. Introduction</b>	<b>73</b>
9.0.1. Examples . . . . .	74
9.1. Graphs . . . . .	74
9.1.1. How to Draw Graphs . . . . .	75
9.1.2. Useful Graphs to Know . . . . .	75
9.2. Level Sets . . . . .	77
9.3. Functions of $\geq 3$ Variables . . . . .	80
9.4. Videos . . . . .	82
<b>10. Partial Derivatives</b>	<b>83</b>
10.1. Geometry of Partial Derivatives . . . . .	83
10.1.1. Implicit Partial Derivatives . . . . .	84
10.1.2. Implicit Differentiation . . . . .	85
10.2. Higher Derivatives . . . . .	86
10.3. Partial Differential Equations . . . . .	87
10.4. Videos . . . . .	89
10.4.1. Calculus Blue . . . . .	89
10.4.2. Khan Academy . . . . .	89
10.4.3. Example Problems: . . . . .	89
<b>11. Linearization</b>	<b>91</b>
11.1. The Fundamental Strategy of Calculus . . . . .	91
11.1.1. Differentiability . . . . .	93
11.2. Differentials . . . . .	94
11.3. Videos . . . . .	95
11.3.1. Calculus Blue: . . . . .	95
11.3.2. Khan Academy . . . . .	95
11.3.3. Example Problems: . . . . .	95
<b>12. The Gradient</b>	<b>97</b>
12.1. Directional Derivatives . . . . .	97
12.2. The Gradient . . . . .	98
12.3. Geometry of the Gradient . . . . .	99
12.3.1. Tangent Planes to Level Sets . . . . .	101
12.4. Videos . . . . .	103
12.4.1. Calculus Blue . . . . .	103
12.4.2. Khan Academy: . . . . .	103

Table of contents

12.4.3. Example Problems . . . . .	104
<b>13. Extrema</b>	<b>105</b>
13.1. Finding Maxima Minima and Saddles . . . . .	109
13.2. Sketching Multivariate Functions . . . . .	109
13.3. Videos . . . . .	111
13.3.1. Calculus Blue . . . . .	111
13.3.2. Khan Academy . . . . .	111
13.3.3. Example Problems . . . . .	111
13.3.4. Optimization Example Problems: . . . . .	112
<b>14. Constrained Optimization</b>	<b>113</b>
14.1. Method I: Reduce Dimension by Substitution . . . . .	113
14.2. Method II: Lagrange Multipliers . . . . .	114
14.3. Optimization and Inequalities: . . . . .	117
<b>IV. Multivariate Integration</b>	<b>119</b>
<b>15. Double Integrals</b>	<b>121</b>
15.1. Riemann Sums and Iterated Integrals . . . . .	121
15.2. Rectangular Domains . . . . .	122
15.3. Variable Boundaries . . . . .	123
15.3.1. Changing the Order of Integration . . . . .	126
15.4. Combining Integrals . . . . .	127
<b>16. Triple Integrals</b>	<b>129</b>
16.1. Different Bounds: . . . . .	129
16.1.1. Boxes . . . . .	129
16.1.2. Variables in One Bound . . . . .	130
16.1.3. Variables in Two Bounds . . . . .	130
16.2. Describing the Bounds: . . . . .	131
16.3. Video Resources . . . . .	131
<b>17. Integrals &amp; Coordinates</b>	<b>133</b>
17.1. Polar Coordinates . . . . .	133
17.2. Cylindrical Coordinates . . . . .	135
17.3. Spherical Coordinates . . . . .	136
17.4. Video Resources . . . . .	137

<b>V. Homework</b>	<b>139</b>
<b>Assignment 1</b>	<b>141</b>
Dimensions . . . . .	141
Vector Addition . . . . .	141
Diagonal of a Cube . . . . .	143
Solutions . . . . .	144
Dimenions . . . . .	144
Vector Addition . . . . .	144
Diagonal of a Cube . . . . .	145
<b>Assignment 2</b>	<b>149</b>
Dot Products and Image Processing . . . . .	149
Planes and Perpendicularity . . . . .	150
Shapes . . . . .	151
Solutions . . . . .	151
Dot Products and Image Processing . . . . .	151
Planes and Perpendicularity . . . . .	152
Shapes . . . . .	153
<b>Assignment 3</b>	<b>155</b>
1. Parametric Curves in the Sciences . . . . .	155
2. Designing Parametric Curves . . . . .	156
3. Finding Intersections . . . . .	157
Solutions . . . . .	158
Parametric Curves in the Sciences . . . . .	158
Designing Parametric Curves . . . . .	158
Finding Intersections . . . . .	160
<b>Assignment 4</b>	<b>163</b>
Problems . . . . .	163
Mission to an Asteroid . . . . .	163
Logo Design, Part I . . . . .	165
Logo Design, Part II . . . . .	167
Solutions . . . . .	170
Mission to an Asteroid . . . . .	170
Logo Design I . . . . .	172
Logo Design II . . . . .	174
<b>Study Guide I</b>	<b>177</b>
Space . . . . .	177
Dimensions . . . . .	177

*Table of contents*

Vectors . . . . .	178
Operations . . . . .	178
Shapes . . . . .	179
Curves . . . . .	179
Parameterizations . . . . .	179
Calculus . . . . .	179
Geometry . . . . .	180
Differential Equations . . . . .	180
<b>Assignment 5</b>	<b>183</b>
Problems . . . . .	183
Reading Contour Plots . . . . .	183
Partial Derivatives . . . . .	184
Graphs of Multivariate Functions . . . . .	185
Solutions . . . . .	186
Reading Contour Plots . . . . .	186
Partial Derivatives . . . . .	186
Graphs of Multivariate Functions . . . . .	187
<b>Reflection &amp; Correction</b>	<b>189</b>
Extra Credit: Exam Correction . . . . .	189
Hold-Me-Accountable: Exam Reflection . . . . .	191
<b>Assignment 6</b>	<b>193</b>
Problems . . . . .	193
Heatshields on Rockets . . . . .	193
Level Sets & The Gradient . . . . .	195
Directional Differentiation . . . . .	195
Solutions . . . . .	196
Heatshields . . . . .	196
Level Sets & The Gradient . . . . .	198
Directional Differentiation . . . . .	200
<b>Assignment 7</b>	<b>203</b>
Problems . . . . .	203
Function Sketching . . . . .	203
Optimization . . . . .	203
Constrained Optimization . . . . .	204
Solutions . . . . .	205
Function Sketching . . . . .	205
Optimization . . . . .	207



*Table of contents*

Constrained Optimization . . . . .	208
<b>Study Guide II</b>	<b>211</b>
Differentiation . . . . .	211
Functions . . . . .	211
Partial Derivatives . . . . .	212
Linearization . . . . .	212
The Gradient . . . . .	212
Extrema . . . . .	212
Optimization . . . . .	212
Integration . . . . .	213
Double Integrals . . . . .	213
<b>Assignment 8</b>	<b>215</b>
Problems . . . . .	215
The Mass of Steel Sheets . . . . .	215
Building Roads . . . . .	216
Calculating Air Pressure . . . . .	217
Solutions . . . . .	219
Mass of Metal Sheets . . . . .	219
Building Roads . . . . .	220
Calculating Air Pressure . . . . .	221
<b>Reflection &amp; Correction II</b>	<b>223</b>
Extra Credit: Exam Correction . . . . .	223
Hold-Me-Accountable: Exam Reflection . . . . .	225
<b>Assignment 9</b>	<b>227</b>
Problems . . . . .	227
Optimizing Volume of a Wineglass . . . . .	227
The Mass of the Earth . . . . .	228
Sailing in the Bay . . . . .	229



# Preface

$$\iint_{\Sigma} \nabla \times \vec{F} \cdot d\vec{S} = \int_{\partial\Sigma} \vec{F} \cdot d\vec{s}$$

This is the set of course notes for my Calculus 3 course (math 211) at the University of San Francisco.



**Part I.**

# **Space**



In this part we acquaint ourselves with the mathematics of  $n$ -dimensional spaces. Such spaces are described using  $n$ -tuples of real numbers

$$(x_1, x_2, \dots, x_n)$$

and are indispensable when discussing positions in real, *physical* space (especially in dimension 3 and 4 for space and spacetime). But they are also indispensable to mathematicians working on other problems, where high dimensional spaces are used to track the behavior of complex objects.

We learn to differentiate between *points* (which measure position) and *vectors* (which give a direction and magnitude), and cover the operations of dot and cross product which are essential to the geometry of vectors.

Finally, we use our knowledge of points and vectors to construct formulas for shapes in 2,3 and higher dimensional spaces; from lines planes and spheres to more complicated objects.





# 1. Dimensions

**(Relevant Section of the Textbook: 12.1 Three Dimensional Coordinate Systems, and 10.3 Polar Coordinates)**

To do calculus in higher dimensions, we need first a precise mathematical language that will allow us to *describe* these spaces. And that language begins with a foundational, but straightforward definition: the  $n$ -tuple.

**Definition 1.1** ( $n$ -Tuple). An  $n$  tuple of real numbers is an ordered list of real numbers. For example, a 2-tuple like  $(3, 7)$  is often called an *ordered pair*. Tuples are sometimes written horizontally and sometimes vertically, depending on convenience. Various styles of brackets are used on tuples, depending on the author and usage. Below are examples of a 3-tuple, a 4-tuple, and a 7-tuple in several styles:

$$\langle 0, -12, 0.3 \rangle \quad \begin{pmatrix} 3 \\ -7 \\ \pi \\ 4 \end{pmatrix} \quad [1, 2, 3, 4, 5, 6, 7]$$

Just as numbers represent a location on the line, tuples can be used to represent locations in space. When we use them as such, we call the entire  $n$ -tuple a *point*, and we call each of the entries a *coordinate*.

**Definition 1.2** (Point). A *point* is an  $n$ -tuple when it is being used to represent a location in space.

You are already familiar with this from single variable calculus, where we use ordered pairs  $(x, y)$  to represent points of the 2-dimensional plane  $\mathbb{R}^2$ . By extension, we can use three-tuples  $(x, y, z)$  to represent points in the physical space around us. But what about even bigger tuples, like the 7-tuple  $[1, 2, 3, 4, 5, 6, 7]$ ? What kind of space does this represent a point in? This is a point in a *seven* dimensional space!

**Definition 1.3.** The dimension of a space is the number of coordinates needed to describe a point in it.

## 1. Dimensions

A plane is 2-dimensional, but so is the surface of a sphere: if your friend called you and gave you two numbers - their latitude and longitude - you could precisely locate them on the Earth's surface.

(37.7749° N, 122.4194° W)

The space around us is three dimensional because if I wanted to direct you to my apartment I would need to give you not only the two street intersections (two numbers, specifying a point on the earth's surface) but also the *floor I live on* (the height above the surface).

(1<sup>st</sup> Street, 3<sup>rd</sup> Ave, 4<sup>th</sup> Floor)

But the space-time we live in is *four-dimensional* because if we wanted to meet for lunch I would need to give you four numbers, my position in space and also when to meet, so that we do not miss each other, you thinking lunch is at 11 and I thinking noon.

(1<sup>st</sup> Street, 3<sup>rd</sup> Ave, 4<sup>th</sup> Floor, 12pm)

Thus, there are direct physical reasons to consider calculus in two, three and four dimensions. And, our best physical theories of the world at human scales (classical mechanics) are written in this language. Understanding weather, planetary motion, fluid flow, and black holes requires a solid grounding in multivariable calculus. But, since the real world is only four dimensional, does that mean there is no need for the calculus of 7, or 13, or 132,234,453 dimensional space?

### 1.1. Higher Dimensions and Configurations

All the spaces we have talked about so far represent *physical space*, but mathematics allows us to be much more general than this. Imagine you are designing a tin can, and you want to start by thinking of the *space of possibilities*: what are all the possible shapes of a cylindrical can? As such a can is fully determined by its radius and its height, we can think of these as being two *coordinates*, and expressing a particular can by an ordered pair  $(r, h)$ . Thus, the *space of possible cans* is 2-dimensional!

What about the space of  $L$ -spaced desks? What is the dimension of this space? This space has 5-dimensions: the length and width of each of the two sides of the desk, and also its height.

$$(\ell_1, \ell_2, w_1, w_2, h)$$

But where really mind-blowing numbers of dimensions begin to arise is in the study of *data*. Imagine you are modeling the conditions in the San Francisco bay, and you take a measurement of the sea height for every square kilometer. The bay has an area of 4000 square kilometers, so this means your datapoint has 4,000 numbers in it! Your approximation to the bay's surface is a point in four thousand dimensional space!

Or, consider an image taken by a digital camera: for simplicity assume the image is in black and white, and 10 megapixels. This means each pixel is determined by a single number (how light or dark the pixel is), and the image has 10 million pixels, so it's encoded by ten million numbers! That means even simple images are worked with mathematically as *points* in a space with millions of dimensions.

## 1.2. Cartesian Coordinates

What do the actual numbers in the  $n$ -tuple mean? In the examples above we have been implicitly using *xyz* or length-width-height coordinates: the first number tells you the distance left/right, the second back/forth and the third up/down. These are called *Cartesian Coordinates* after the mathematician-philosopher Rene Descartes. While this sort of thinking comes from *physical* space, it is useful to help give a concrete picture even to *non-physical spaces* such as the space of soup cans, or the space of images.

**Definition 1.4** (Cartesian Coordinates). Cartesian coordinates starts by choosing  $n$  perpendicular lines in  $n$  dimensional space: for example the  $x$  and  $y$  axes in the plane, or the  $x, y, z$  axes in  $\mathbb{R}^3$ . A point in space is given coordinates  $(a, b, c)$  if it lies at distance  $a$  along the first axis,  $b$  along the second, and  $c$  along the third.

Here's an animation showing a point in 3D space, and its components along the  $x, y$  and  $z$  axes.

<https://stevejtrettel.site/code/2023/vector-components>

Perhaps the most famous theorem of geometry is the Pythagorean Theorem, which tells us how to compute distance in the cartesian coordinates on the plane:

**Theorem 1.1.** *The distance of the point  $(a, b)$  from the point  $(0, 0)$  in the plane is*

$$\text{dist} = \sqrt{a^2 + b^2}$$

## 1. Dimensions

Once we know this theorem is true in  $\mathbb{R}^2$  (thanks, Pythagoras!) we can use this proof to extend it to higher dimensions!

**Theorem 1.2.** In  $\mathbb{R}^3$ , the distance of  $(x, y, z)$  from the origin is

$$\text{dist} = \sqrt{x^2 + y^2 + z^2}$$

In general, if  $(x_1, \dots, x_n)$  is a point in  $\mathbb{R}^n$ , its distance from the origin is

$$\text{dist} = \sqrt{\sum_{i=1}^n x_i^2}$$

Given this description of distance, we can give a precise description of circles and spheres: a circle is the set of points which are all a fixed distance (the radius) from a fixed point (the center). The *unit circle* is the set of points distance 1 from the origin. Similarly, the unit sphere is the set of points distance 1 from the origin in 3-dimensional space.

In more generality, we can define a *hypersphere* in any higher dimension the same way: by taking the set of points which are a fixed distance from the origin in that space!

**Definition 1.5.** The unit circle (sometimes called the unit 1-sphere) is the set of points in  $\mathbb{R}^2$  given by

$$x^2 + y^2 = 1$$

The unit sphere (sometimes called the unit 2-sphere) is the set of points in  $\mathbb{R}^3$  satisfying

$$x^2 + y^2 + z^2 = 1$$

The unit 3-sphere is the set of points in *four dimensional space* satisfying

$$x^2 + y^2 + z^2 + w^2 = 1$$

And so on...

**Exercise 1.1.** Why is the circle called a 1-sphere, and the sphere in 3D space called the 2-sphere?

We can also use cartesian coordinates to describe other simple shapes, like lines. In  $\mathbb{R}^2$  with cartesian coordinates, the  $x$ -axis is given by the set of points  $\{(x, 0)\}$ : that is, every point on the  $x$ -axis has  $y$ -coordinate equal to zero.

**Definition 1.6** (Coordinate Axes and Planes). The  $x$  axis in the plane is given by the equation  $y = 0$ , and the  $y$  axis by the equation  $x = 0$ . Similarly in  $\mathbb{R}^3$ , the  $xy$  plane is given by the equation  $z = 0$ , the  $yz$  plane by the equation  $x = 0$ , and the  $xz$  plane by the equation  $y = 0$ .

A point can be *projected* onto a coordinate axis or plane by setting that coordinate to zero. The resulting point is the *closest* point on that line or plane to the original point. This makes it a relatively straightforward calculation to find the distance from a point to a coordinate axis/plane!

Here's some practice problems:

### 1.3. Other Coordinate Systems

While the majority of our class will occur in Cartesian coordinates as they are the first coordinate system everyone must master, we will at times consider a couple of other interpretations of  $n$ -tuples, which make describing certain systems with circular or spherical symmetry easier. The first of these may already be familiar from earlier calculus classes: polar coordinates on the plane.

**Definition 1.7** (Polar Coordinates). A point  $(r, \theta)$  in polar coordinates on the plane, for  $r > 0$  and  $\theta \in [0, 2\pi)$  represents the point which lies at a distance  $r$  from the origin, and makes an angle of  $\theta$  with the positive  $x$ -axis.

From this description we can convert a point in polar coordinates to cartesian coordinates  $(x, y)$  using trigonometry:

**Definition 1.8.** The conversion from polar coordinates  $(r, \theta)$  to cartesian coordinates  $(x, y)$  is given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix}$$

Using polar coordinates simplifies many things when circles are involved: for example, the equation of the unit circle  $x^2 + y^2 = 1$  becomes the much simpler equation  $r = 1$  in polar coordinates!

Using polar coordinates for  $x, y$  in the 3-dimensional space  $(x, y, z)$  gives a coordinate system called *cylindrical coordinates*.

## 1. Dimensions

**Definition 1.9** (Cylindrical Coordinates). A 3-tuple  $(r, \theta, z)$  represents a point in  $\mathbb{R}^3$  using cylindrical coordinates where the position in the  $xy$  plane is given by the polar coordinates  $(r, \theta)$  and the height above the  $xy$  plane is given by  $z$ . The conversion to cartesian coordinates is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ z \end{pmatrix}$$

Finally we will come across a third coordinate system in this class: *spherical coordinates*. This represents 3-dimensional space starting with a collection of concentric spheres. Don't worry too much about this now, we will come back to it in some weeks! I've only placed it here for your future reference.

**Definition 1.10** (Spherical Coordinates). A 3-tuple  $(\rho, \theta, \phi)$  represents a point in  $\mathbb{R}^3$  using spherical coordinates where  $\rho$  is the distance from the origin,  $\theta$  is the angle with the positive  $x$  axis (as in polar coordinates) and  $\phi$  is the angle with the positive  $z$ -axis. The conversion to cartesian coordinates is given by

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \rho \cos \theta \sin \phi \\ \rho \sin \theta \sin \phi \\ \rho \cos \phi \end{pmatrix}$$

## 1.4. Videos

### 1.4.1. Cartesian Coordinates

<https://youtu.be/iBgOoaeLUcM>

### 1.4.2. Higher Dimensions

<https://youtu.be/yjUy9kevd3Y>

<https://youtu.be/qo0KKSXWW4E>

### 1.4.3. Distances

[https://youtu.be/GJDi4\\_Af\\_II](https://youtu.be/GJDi4_Af_II)

## 2. Vectors

(Relevant Section of the Textbook: 12.2 Vectors)

We have talked about one fundamental use of  $n$ -tuples of real numbers: describing *positions* in space. But they also play a foundational role in the theory of *vectors*, which help us measure not *locations* but *directions*.

**Definition 2.1** (Vector). A vector is a *directed line segment*, an object that stores both a *length* (called its *magnitude*), and a *direction*. You may draw a vector as a directed line segment, or a little arrow in space.

**Example 2.1** (Points vs Vectors). Which of the following quantities are positions? Which are vectors?

- Where I parked my car.
- The wind hitting me in the face
- The location of mars in the solar system.
- The velocity of mars in the solar system.
- Gravity's acceleration

**Exercise 2.1.** Come up with some of your own scenarios that are measured using: 2-d points. 3-d vectors. 4d points, 4d vectors.

To work with vectors, we need a means of writing them down using numbers. One idea is to use the same *Cartesian coordinate system* discussed in the previous chapter, to express a vector in *components*.

**Definition 2.2** (Vector Notation). To help avoid confusing *points* and *vectors*, we will try to use different notations for the two. For a point, we use a simple letter, like  $p$  whereas for a vector we either make it bold like  $\mathbf{u}$  or decorate it with an arrow,  $\vec{u}$ .

When using cartesian coordinates, we write a point inline with round brackets,  $p = (1, 2, 3)$  whereas for a vector we use angle brackets,  $\vec{u} = \langle 1, 2, 3 \rangle$ . We also sometimes

## 2. Vectors

write a vector as a *column* of numbers, instead of a row to save space, or for other stylistic reasons: in this case we just use round brackets for ease of typesetting.

$$\vec{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

**Definition 2.3** (The Zero Vector). The zero vector in an  $n$  dimensional space is the vector with no magnitude and no length. In coordinates, this is the  $n$ -tuple of all zeroes:  $\langle 0, 0, 0 \rangle$

### 2.1. Arithmetic of Vectors

**Definition 2.4** (Vector Addition). Vector addition is defined to give the *combined effect* of two vectors: if  $\vec{u}$  and  $\vec{v}$  are vectors,  $\vec{u} + \vec{v}$  is defined geometrically by the diagonal of the parallelogram with sides  $\vec{u}$  and  $\vec{v}$ . This is equivalent to the vector formed by stacking  $\vec{u}$  and  $\vec{v}$  on one another head-to-tail, in either order.

In cartesian coordinates, this is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} x + a \\ y + b \\ z + c \end{pmatrix}$$

PICTURE

Vector addition can be used to describe complicated motion in terms of simpler pieces. Indeed, this idea was used by the ancient greeks in their planetary model, where the complicated motion of objects in the heavens was modeled as a combination of various circular motions (or epicycles). Below is an animation showing their model of Mars' motion about the earth, decomposed in terms of a sum of three circles.

<https://stevejtrettel.site/code/2020/epicycle-mars/>

**Definition 2.5** (Scalar Multiplication). If  $\vec{u}$  is a vector and  $c$  is a number, the vector  $c\vec{u}$  is defined to be the vector pointing in the same direction as  $\vec{u}$ , but  $c$  times as long. In Cartesian coordinates,

$$c\langle u_1, u_2, u_3 \rangle = \langle cu_1, cu_2, cu_3 \rangle$$

Because we think of real numbers as being the kind of things that can *scale* vectors, we often call them **scalars**.



**Definition 2.6** (Linear Combination). A linear combination of a list of vectors is a new vector made by taking a sum of scalar multiples of the original list.

For example, if  $\vec{u} = \langle 1, 2 \rangle$  and  $\vec{v} = \langle 3, 4 \rangle$ , then the following is a linear combination:

$$\vec{w} = 7 \begin{pmatrix} 1 \\ 2 \end{pmatrix} - 2 \begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 7 - 6 \\ 14 - 8 \end{pmatrix} = \langle 1, 6 \rangle$$

In this calculation you saw a bit of vector arithmetic: it works just like the arithmetic of numbers, one coordinate at a time.

**Theorem 2.1** (Vector Arithmetic). Let  $\vec{u}, \vec{v}$  and  $\vec{w}$  be vectors, and  $c, k$  be scalars. Then:

$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \quad \vec{u} + (\vec{v} + \vec{w}) = (\vec{u} + \vec{v}) + \vec{w}$$

$$\vec{u} + \vec{0} = \vec{u} \quad \vec{u} + (-\vec{u}) = \vec{0}$$

$$c(\vec{u} + \vec{v}) = c\vec{u} + c\vec{v} \quad (c + k)\vec{u} = c\vec{u} + k\vec{u}$$

$$(ck)\vec{u} = c(k\vec{u}) \quad 1\vec{u} = \vec{u}$$

## 2.2. Coordinate Bases

Cartesian coordinates are built from a collection of perpendicular axes. Each of these axes has a *direction* that we call a *standard basis direction*

**Definition 2.7** (Standard Basis). For  $\mathbb{R}^2$  the standard basis vectors are the vectors  $\langle 1, 0 \rangle$  and  $\langle 0, 1 \rangle$ , pointing along the positive direction of the  $x$  and  $y$  axes.

For  $\mathbb{R}^3$ , the standard basis vectors are the vectors  $\langle 1, 0, 0 \rangle$ ,  $\langle 0, 1, 0 \rangle$  and  $\langle 0, 0, 1 \rangle$  pointing along the positive direction of the  $x$ ,  $y$  and  $z$  axes respectively.

In general  $n$ -dimensional space, the  $n$  basis vectors are the vectors which have all 0s as coordinates except a single 1. The vector whose 1 is in the  $i^{\text{th}}$  coordinate is called the  $i^{\text{th}}$  *basis vector*.

## 2. Vectors

For example,  $\langle 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0 \rangle$  is the 4<sup>th</sup> standard basis vector of 12-dimensional space. In two and three dimensions we give each of the bases a unique letter to aid in readability, instead of dealing with messy subscripts for only a handful of symbols.

**Definition 2.8** (Standard Basis in  $\mathbb{R}^2$  and  $\mathbb{R}^3$ ). In  $\mathbb{R}^2$  we write the standard basis vectors as

$$\hat{i} = \langle 1, 0 \rangle \quad \hat{j} = \langle 0, 1 \rangle$$

In  $\mathbb{R}^3$  we continue this, writing

$$\hat{i} = \langle 1, 0, 0 \rangle \quad \hat{j} = \langle 0, 1, 0 \rangle \quad \hat{k} = \langle 0, 0, 1 \rangle$$

We can use these standard basis vectors to express any vector in space. For example, in  $\mathbb{R}^3$  every vector is some amount in the  $\hat{i}$  direction, some amount in the  $\hat{j}$  direction, and some amount in the  $\hat{k}$  direction. This means we can write any vector as a *linear combination* of these:

$$\vec{u} = x\hat{i} + y\hat{j} + z\hat{k}$$

<https://stevejtrettel.site/code/2023/vector-components>

## 2.3. Magnitude and Direction Information

One common use for vectors is to give *directions* to get from one point to another: that is, given points  $p, q$  in space, we want a vector starting at  $p$  and ending at  $q$ .

PIC

This vector encodes the magnitude and direction information of “if you are at  $p$  and you walk this amount in this direction, you’ll arrive at  $q$ ”. We can construct such a vector

**Definition 2.9** (Vector from Two Points). The *displacement vector* from a point  $p = (p_1, p_2, p_3)$  to a point  $q = (q_1, q_2, q_3)$  is

$$\vec{d} = q - p = \begin{pmatrix} q_1 - p_1 \\ q_2 - p_2 \\ q_3 - p_3 \end{pmatrix}$$

And, analogously in other dimensions.

Thus, the vector  $\langle x, y, z \rangle$  is the displacement vector from the *origin* to the point  $(x, y, z)$ . Its length (or magnitude) is just the distance from the origin to its other endpoint, which we know from the pythagorean theorem.

**Definition 2.10** (Magnitude). The magnitude of the vector  $\vec{u} = \langle u_1, u_2, u_3 \rangle$  is given by the pythagorean theorem:

$$\|\vec{u}\| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

And, analogously in other dimensions.

A vector of length 1 is called a *unit vector*. We think of these as measuring *purely direction* just as we think of numbers as measuring *purely length*.

**Definition 2.11** (Unit Vector in Given Direction). If  $\vec{v}$  is a nonzero vector, the unit vector in direction  $\vec{v}$  is denoted  $\hat{v}$ , and is calculated by dividing  $\vec{v}$  by its own magnitude:

$$\hat{v} := \frac{1}{\|\vec{v}\|} \vec{v}$$

**Exercise 2.2** (Unit Vectors in Given Directions). Find a unit vector in the direction of  $\langle 1, 2, 3, 4 \rangle$ .

**Exercise 2.3** (Unit Vectors in Given Directions). Find a vector of length 2 in the direction of  $\langle 1, 1, 1, 1, 1, 1, 1, 1 \rangle$ . *Hint: find a unit vector in this direction. What happens to its length if you scalar multiply it by 2?*

## 2.4. Videos

[https://youtu.be/fNk\\_zzaMoSs](https://youtu.be/fNk_zzaMoSs)

<https://youtu.be/etsFVs354GM>

[https://youtu.be/3V\\_3dnO-0lo](https://youtu.be/3V_3dnO-0lo)

<https://youtu.be/F6iTnOoJ9as>

## 2. Vectors

### 2.4.1. Video Tutorial Series

Here's a short series of videos going through the different vector properties discussed above:

<https://youtu.be/51vgIfdBIAk>

<https://youtu.be/MoHMw0ZO7bU>

<https://youtu.be/lulSApFPw1M>

<https://youtu.be/MpN8BIci-Ys>

<https://youtu.be/DXB1PWq8Dg0>

## 3. Operations

(Relevant Sections of the Textbook: 12.3 The Dot Product, and 12.4 The Cross Product)

### 3.1. The Dot Product

The dot product is an operation which takes in two vectors and outputs a single number. We will use it as a tool to measure several things, but it's perhaps easiest to learn by just diving right in with the definition.

**Definition 3.1** (Dot Product). If  $\vec{u} = \langle u_1, u_2, u_3 \rangle$  and  $\vec{v} = \langle v_1, v_2, v_3 \rangle$  then the dot product of  $\vec{u}$  and  $\vec{v}$  is the scalar

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

And similarly in other dimensions, if  $\vec{u}, \vec{v}$  are vectors in  $n$  dimensions, then

$$\vec{u} \cdot \vec{v} = \sum_{i=1}^n u_i v_i$$

For example, the dot product of  $\langle 1, 2 \rangle$  and  $\langle 3, 4 \rangle$  is  $1 \times 3 + 2 \times 4 = 3 + 8 = 11$ , and the dot product of  $\hat{i} - \hat{j}$  and  $2\hat{i} - \hat{j} + 3\hat{k}$  can be computed by either (1) converting to coordinate notation, or (2) pairing up coefficients and multiplying.

$$(\hat{i} - \hat{j}) \cdot (2\hat{i} - \hat{j} + 3\hat{k}) = (1 \times 2) + (-1 \times -1) + (0 \times 3) = 3$$

We will unpack a lot more of the geometry hidden inside of this simple definition soon, but a first thing to notice is that the *magnitude* of a vector can be recovered from its dot product with itself.

**Theorem 3.1.** *If  $\vec{u}$  is any vector, then the magnitude of  $\vec{u}$  can be calculated via*

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$$

### 3. Operations

**Exercise 3.1.** Check that this works for the vector  $\langle x, y, z \rangle$  in  $\mathbb{R}^3$ .

The dot product is built out of the multiplication and addition of ordinary numbers, so it also inherits a lot of their algebraic properties:

**Theorem 3.2** (Properties of the Dot Product). *If  $\vec{u}, \vec{v}$  and  $\vec{w}$  are vectors and  $c, k$  are scalars then*

$$\vec{0} \cdot \vec{u} = \vec{0} \quad \vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$$

$$\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w} \quad c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$$

#### 3.1.1. Measuring Angles

We've already seen that the dot product of a vector with itself measures the (square of the) magnitude of that vector. But dot products are also able to recover *direction* information as well.

**Theorem 3.3** (Angles and the Dot Product). *If  $\vec{u}, \vec{v}$  are vectors, their dot product is related to the angle  $\theta$  between them via*

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$$

*Re-arranging this, we get a formula that computes the angle between two vectors using only vector operations!*

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{u}||\vec{v}|}$$

**Exercise 3.2.** What is the angle between  $\langle 1, 2 \rangle$  and  $\langle 3, 4 \rangle$  in radians?

What is the angle between  $\hat{i} + \hat{j}$  and  $\hat{j}$  in degrees?

**Definition 3.2** (Orthogonality). Two vectors are called orthogonal if the angle between them is 90 degrees, or  $\pi/2$  radians. Since  $\cos(\pi/2) = 0$ , this means that two vectors are orthogonal if and only if their dot product is zero.

Note that the zero vector dotted with any other vector always gives zero, so we say the zero vector is *orthogonal* to every other vector.

### 3.1.2. Projections

One very useful application of the dot product is to help measure “how much of vector  $\vec{v}$  is pointed in direction  $\hat{u}$ ?”

The *scalar projection* of  $\vec{v}$  onto the unit vector  $\hat{u}$  is the dot product  $\vec{v} \cdot \hat{u}$ . If  $\vec{u}$  is not a unit vector to start with, we first make it into a unit vector by dividing by its magnitude, to find the scalar projection:  $\vec{v} \cdot \hat{u} = \vec{v} \cdot \frac{\vec{u}}{|\vec{u}|}$

**Definition 3.3** (Scalar Projection). The scalar projection of  $\vec{v}$  onto  $\vec{u}$  is

$$\text{comp}_{\vec{u}}(\vec{v}) = \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|}$$

**Exercise 3.3** (Scalar Projection). What is the scalar projection of  $\langle 1, 2 \rangle$  onto  $\langle 3, 4 \rangle$ ?

What is the scalar projection of  $\langle 4, 3, -2 \rangle$  onto  $\langle 0, 1, 0 \rangle$ ?

This tells us *how much* of a vector is pointed in a given direction, and so the answer is a scalar, or number. Oftentimes it is useful to compute a *vector* from this, whose direction is in the direction of  $\vec{u}$ , and magnitude is the scalar we just computed. This vector can be thought of as the projection of  $\vec{v}$  onto  $\vec{u}$ , or the *shadow* of  $\vec{v}$  on the line spanned by  $\vec{u}$ .

PICTURE

**Definition 3.4** (Vector Projection). The *vector projection* of a vector  $\vec{v}$  onto a vector  $\vec{u}$  is the *scalar projection* times the unit vector in direction  $\vec{u}$ . In symbols:

$$\begin{aligned} \text{proj}_{\vec{u}}(\vec{v}) &= \text{comp}_{\vec{u}}(\vec{v}) \frac{\vec{u}}{|\vec{u}|} \\ &= \left( \frac{\vec{v} \cdot \vec{u}}{|\vec{u}|} \right) \frac{\vec{u}}{|\vec{u}|} \\ &= \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} \end{aligned}$$

**Exercise 3.4** (Vector Projection). Find the vector projection of  $\langle 1, 1, 2 \rangle$  onto the vector  $\langle -2, 3, 1 \rangle$ .

### 3. Operations

#### 3.1.3. Standard Basis

**Theorem 3.4.** If  $\vec{v} = \langle a, b, c \rangle$  is a vector, its scalar projections onto the three standard basis vectors  $\hat{i}, \hat{j}, \hat{k}$  are

$$\text{comp}_{\hat{i}}(\vec{v}) = \vec{v} \cdot \hat{i} = a$$

$$\text{comp}_{\hat{j}}(\vec{v}) = \vec{v} \cdot \hat{j} = b$$

$$\text{comp}_{\hat{k}}(\vec{v}) = \vec{v} \cdot \hat{k} = c$$

This confirms our notions of “amount” and “angle” make sense with our original interpretation of the vector  $\langle a, b, c \rangle$  as being an amount  $a$  in the direction  $\hat{i}$ ,  $b$  in the direction  $\hat{j}$ , and  $c$  in the direction  $\hat{k}$ .

**Theorem 3.5 (Direction Angle).** The angles  $\alpha, \beta, \gamma$  a vector  $\vec{v} = \langle a, b, c \rangle$  makes with the direction of the standard basis  $\hat{i}, \hat{j}, \hat{k}$  respectively are

$$\cos \alpha = \frac{\vec{v} \cdot \hat{i}}{\|\vec{v}\|} = \frac{a}{\|\vec{v}\|}$$

$$\cos \beta = \frac{\vec{v} \cdot \hat{j}}{\|\vec{v}\|} = \frac{b}{\|\vec{v}\|}$$

$$\cos \gamma = \frac{\vec{v} \cdot \hat{k}}{\|\vec{v}\|} = \frac{c}{\|\vec{v}\|}$$

### 3.2. Cross Product

In two dimensions, given one vector  $\vec{v} = \langle a, b \rangle$  it is easy to find an orthogonal vector to it:  $\vec{v}^\perp = \langle b, -a \rangle$  for example.

In 3D, the analogous problem is more difficult: given two vectors, how do we find a third vector that is orthogonal to them *both at once*? A nice solution to this is given by the **cross product**.

**Definition 3.5 (Cross Product).** The cross product of  $\vec{u} = \langle u_x, u_y, u_z \rangle$  and  $\vec{v} = \langle v_x, v_y, v_z \rangle$  is

$$\vec{u} \times \vec{v} = \langle u_y v_z - u_z v_y, u_x v_z - u_z v_x, u_x v_y - u_y v_x \rangle$$

<https://stevejtrettel.site/code/2023/cross-product>



### 3.2. Cross Product

**Exercise 3.5** (Orthogonality of the Cross Product). Check that if  $\vec{u} = \langle a, b, c \rangle$  and  $\vec{v} = \langle x, y, z \rangle$  are two vectors, that  $\vec{u} \times \vec{v}$  is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .

While there are many vectors orthogonal to  $\vec{u}$  and  $\vec{v}$ , this particular choice has some very nice mathematical properties. In particular, it gets its name because it acts algebraically a lot like multiplication:

**Theorem 3.6** (Properties of the Cross Product). Let  $\vec{u}, \vec{v}, \vec{w}$  be vectors, and  $k$  a scalar. Then

$$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

$$(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$$

$$(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$$

However, the cross product has two **very important differences** from regular multiplication: the order matters!

**Theorem 3.7** (Non-Commutativity of the Cross Product). If  $\vec{u}$  and  $\vec{v}$  are vectors, then

$$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

Not only does the order that you place the vectors in the product matter, but if you are doing more than one cross product, the order in which you perform them matters as well!

**Theorem 3.8** (Non-Associativity of the Cross Product). If  $\vec{u}, \vec{v}$  and  $\vec{w}$  are vectors, then

$$(\vec{u} \times \vec{v}) \times \vec{w} \neq \vec{u} \times (\vec{v} \times \vec{w})$$

#### 3.2.1. Computing the Cross Product

**Definition 3.6** ( $2 \times 2$  determinants). The determinant of a  $2 \times 2$  matrix is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

### 3. Operations

Using this notation, we can recast the definition of the cross product from Definition 3.5 to look like

$$\vec{u} \times \vec{v} = \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} \hat{i} - \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} \hat{j} + \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \hat{k}$$

Note the *minus sign* on the second term (compare with Definition 3.5 to see where a  $-1$  was factored out). This formula can in turn be written even more compactly using the definition of a  $3 \times 3$  determinant:

**Definition 3.7** ( $3 \times 3$  determinants). The determinant of a  $3 \times 3$  matrix is

$$\begin{vmatrix} x & y & z \\ a & b & c \\ d & e & f \end{vmatrix} = x \begin{vmatrix} b & c \\ e & f \end{vmatrix} - y \begin{vmatrix} a & c \\ d & f \end{vmatrix} + z \begin{vmatrix} a & b \\ d & e \end{vmatrix}$$

Putting it all together,

**Definition 3.8.** The cross product of  $\vec{u} = \langle u_x, u_y, u_z \rangle$  and  $\vec{v} = \langle v_x, v_y, v_z \rangle$  is

$$\begin{aligned} \vec{u} \times \vec{v} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} \\ &= \begin{vmatrix} u_y & u_z \\ v_y & v_z \end{vmatrix} \hat{i} - \begin{vmatrix} u_x & u_z \\ v_x & v_z \end{vmatrix} \hat{j} + \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \hat{k} \end{aligned}$$

While the cross product is specific to three dimensions, later in the course we will sometimes apply it to 2-dimensional vectors, where we think of a vector  $\vec{v} = \langle x, y \rangle$  in  $\mathbb{R}^2$  as being the vector  $\vec{v} = \langle x, y, 0 \rangle$  in  $\mathbb{R}^3$ . In this case, the cross product of two planar vectors has a rather simple formula (for instance, we know it must point directly along the  $z$ -axis!)

**Definition 3.9** (Planar Cross Product). If  $\vec{u} = \langle a, b \rangle$  and  $\vec{v} = \langle c, d \rangle$  are two vectors in  $\mathbb{R}^2$ , their cross product, *when thought of as vectors in the  $xy$  plane of  $\mathbb{R}^3$*  is

$$\vec{u} \times \vec{v} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \hat{k} = \langle 0, 0, ad - bc \rangle$$

### 3.2.2. Geometry of the Cross Product

The main property of the cross product is that it's a third vector that is orthogonal to the two vectors you start with. If you did Exercise 3.5, you've already proven this theorem:

**Theorem 3.9** (Orthogonality of the Cross Product). *The vector  $\vec{u} \times \vec{v}$  is orthogonal to both  $\vec{u}$  and  $\vec{v}$ .*

However there is an entire line of vectors orthogonal to  $\vec{u}$  and  $\vec{v}$ . Which of these is the cross product? We need to understand its *magnitude* and its *direction*.

**Theorem 3.10.** *The magnitude of the cross product  $\vec{u} \times \vec{v}$  is given in terms of the magnitudes of  $\vec{u}$  and  $\vec{v}$ , and the angle  $\theta$  between them:*

$$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin \theta$$

This formula, *base times height times the sine of the angle between them* may be familiar as the area of a parallelogram. That gives an even more useful interpretation of the cross product's length:

**Theorem 3.11.** *The magnitude of the cross product  $\vec{u} \times \vec{v}$  is the area of the parallelogram spanned by  $\vec{u}$  and  $\vec{v}$ .*

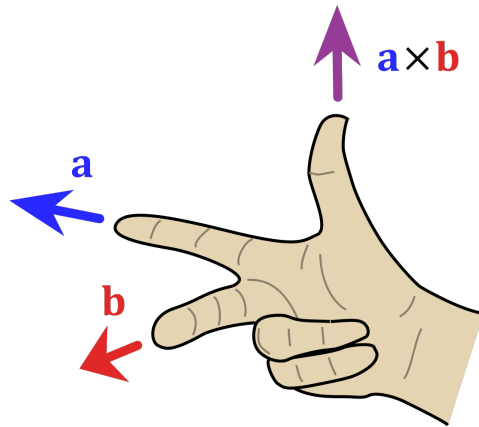
This has an immediate corollary: if  $\vec{u}$  and  $\vec{v}$  are parallel the parallelogram they span collapses onto a line. And lines have zero area, so the cross product must have zero length! That is, it must be the vector of all zeroes.

**Corollary 3.1.** *If  $\vec{u}$  and  $\vec{v}$  are parallel, then their cross product is the zero vector.*

This gives us the *magnitude* information, but what about the direction? There are two possible directions a vector could point if it must be perpendicular to the plane containing  $\vec{u}$  and  $\vec{v}$

**Theorem 3.12** (The Right Hand Rule). *The right hand rule is a mnemonic to help remember the direction of the cross product. If you take your right hand and align your palm with the vector  $\vec{u}$  and then curl your fingers towards the vector  $\vec{v}$ , your thumb will point in the direction  $\vec{u} \times \vec{v}$ .*

### 3. Operations



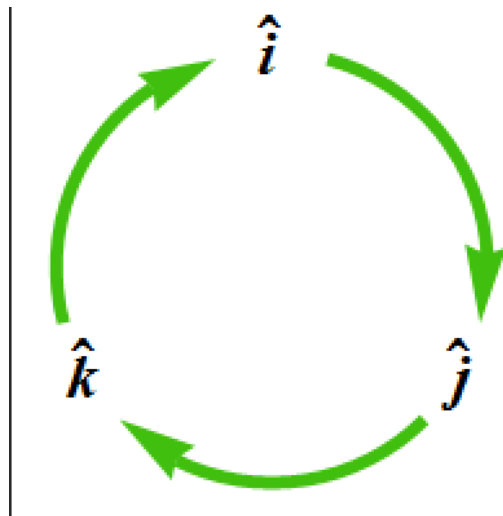
#### 3.2.3. The Standard Basis

It's often useful to know the value of the cross product on the standard basis vectors, to speed up some computations and avoid the lengthy formula in Definition 3.8.

**Theorem 3.13.**

$$\hat{i} \times \hat{j} = \hat{k} \quad \hat{j} \times \hat{k} = \hat{i} \quad \hat{k} \times \hat{i} = \hat{j}$$

This is helpfully illustrated by a diagram:



### 3.3. The Triple Product

Recalling that the cross product changes sign when you reverse the order of the inputs, its possible to compute the rest of the possible stanard basis products:

$$\hat{i} \times \hat{j} = \hat{k} \implies \hat{j} \times \hat{i} = -\hat{k}$$

In terms of the diagram, this means if you read “backward” along an arrow, you insert a minus sign.

### 3.3. The Triple Product

If you have three vectors, its possible to combine the dot and cross product to get a single number: take the cross product of two of them to get another vector, then dot that result with the third.

**Definition 3.10** (Triple Product). The scalar triple product of the vectors  $\vec{u}$ ,  $\vec{v}$  and  $\vec{w}$  is defined by

$$\vec{u} \cdot (\vec{v} \times \vec{w})$$

Writing out the cross product using Definition 3.8, we see that this is actually a big  $3 \times 3$  determinant:

**Theorem 3.14** (Triple Product). *The triple product of vectors  $\vec{u}$ ,  $\vec{v}$ ,  $\vec{w}$  is equal to the  $3 \times 3$  determinant*

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

Just as the  $2 \times 2$  determinant measures the *area of a parallelogram*, this  $3 \times 3$  determinant measures the *volume of a parallepiped*

**Theorem 3.15** (Triple Products & Volume). *The magnitude of the triple product of  $\vec{u}$ ,  $\vec{v}$ ,  $\vec{w}$  is the volume of the parallepiped spanned by these vectors.*

### 3. Operations

## 3.4. Videos

### 3.4.1. Dot Products

From the “Calculus Blue” series by Prof Robert Ghrist.

<https://youtu.be/kw-TTdfJwY0>

<https://youtu.be/nRh8XqWEoj4>

<https://youtu.be/LxSMhIUaIc4>

<https://youtu.be/Dsa2MX2grMg>

A video reviewing the basic definition of the dot product.

<https://youtu.be/Tfuu7iSxGIA>

The following two videos review the theory and an example for using the dot product to find the angle between vectors.

<https://youtu.be/Tfuu7iSxGIA>

<https://youtu.be/4WxniMJYySc>

A short video on the definition of orthogonality.

<https://youtu.be/tGYvaabMbYA>

Real world applications of the dot product

<https://youtu.be/TBpDMLCC2uY?si=8GL1huhqLnIDowQy>

### 3.4.2. Cross Products

[https://youtu.be/\\_tXqoAehVR0](https://youtu.be/_tXqoAehVR0)

<https://youtu.be/-734RN3BqPk>

<https://youtu.be/UAAQlqMCc8c>

### 3.4.3. Triple Products

<https://youtu.be/McWNGB1USQE>

## 4. Shapes

(Relevant Sections of the Textbook: 12.5 Lines & Planes, and 12.6 Cylinders and Quadric Surfaces)

Lines and planes are given by *linear* equations: involving only the coordinate variables ( $x, y, z$ , etc), constants, and addition.

### 4.1. Lines

**Definition 4.1** (Implicit Lines in  $\mathbb{R}^2$ ). An implicit line in the plane is an equation of the form

$$ax + by = c$$

. When  $b \neq 0$  this can be put into  $y = mx + b$  form as  $y = -\frac{a}{b}x + \frac{c}{b}$ .

**Theorem 4.1** (Normal Direction to a Line). *The implicit line  $ax + by = c$  is orthogonal to the vector  $\vec{n} = \langle a, b \rangle$ .*

How can we confirm this? Find *two points* along the line, subtract them to get their *direction vector*, then take the dot product of this with  $\langle a, b \rangle$  - show its zero!

Finding the direction of a line by finding two points on it is tedious, so luckily once we know the above fact we don't need to do this any more! Its easy to figure out the *direction* of an implicit line: if its orthogonal to  $\langle a, b \rangle$ , then it points in the direction  $\langle -b, a \rangle$  (recall this vector is itself orthogonal to  $\langle a, b \rangle$ ).

**Definition 4.2** (Parametric Lines). A parametric line is a function of the form

$$\ell(t) = p + t\vec{v}$$

This passes through the point  $p$  and is in the direction  $\vec{v}$ .

<https://stevejtrettel.site/code/2023/parameterized-line>

## 4. Shapes

**Theorem 4.2** (Line Segment Between Two Points). *If  $\vec{p}$  and  $\vec{q}$  are two points in space, the line segment between them can be parameterized by the following equation for  $t \in [0, 1]$ .*

$$\begin{aligned}\vec{\ell}(t) &= \vec{p} + t(\vec{q} - \vec{p}) \\ &= (1 - t)\vec{p} + t\vec{q}\end{aligned}$$

### 4.2. Planes

**Definition 4.3** (Implicit Planes I). A plane through the point  $\vec{p} = \langle x_0, y_0, z_0 \rangle$  with normal vector  $\vec{n} = \langle a, b, c \rangle$  is determined by the scalar equation

$$\begin{aligned}\vec{n} \cdot (\langle x, y, z \rangle - \vec{p}) &= 0 \\ a(x - x_0) + b(y - y_0) + c(z - z_0) &= 0\end{aligned}$$

Distributing and collecting all the constants on the right hand side, we see planes are given by simple, *linear* equations in three variables.

**Definition 4.4** (Implicit Planes II). A plane in  $\mathbb{R}^3$  is specified by a scalar equation of the form

$$ax + by + cz = d$$

The original derivation also allows us to easily read off the normal vector to a plane:

**Theorem 4.3** (Normal Direction to a Plane). *The implicit plane  $ax + by + cz = d$  is orthogonal to the vector of coefficients  $\vec{n} = \langle a, b, c \rangle$*

To write down a parametric line, we chose a point  $p$  in space, and a direction  $\vec{v}$ . We then added scaled versions of  $\vec{v}$  to  $p$ , which traced out a line. We can do something analogous with a plane, except we pick two direction vectors, and have *two* scaling parameters

**Definition 4.5** (Parametric Planes). A parametric plane through the point  $p$  containing the vectors  $\vec{u}$  and  $\vec{v}$  is given by the function

$$P(s, t) = p + s\vec{u} + t\vec{v}$$

<https://stevejtrettel.site/code/2023/parametric-plane>

What is the normal vector to a parametric plane? We know two vectors on the plane  $\vec{u}$  and  $\vec{v}$ , so their cross product must be orthogonal to the plane.



### 4.3. Circles and Spheres

**Theorem 4.4** (Normal to a Parametric Plane). *If  $P(s, t) = p + s\vec{u} + t\vec{v}$  is a parametric plane, the vector*

$$\vec{n} = \vec{u} \times \vec{v}$$

*is a normal vector to it.*

**Exercise 4.1** (Implicit Equation from a Parametric Plane). Consider the following parametric plane:

$$P(s, t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + s \begin{pmatrix} -1 \\ 3 \end{pmatrix} + t(1, -1)$$

What is a point that it passes through? What is its normal vector? What's an implicit equation for this plane?

## 4.3. Circles and Spheres

**Definition 4.6** (Circle). The circle  $C$  of radius  $R$  centered at a point  $p$  in the plane is the set of all points which lie at distance  $R$  from  $p$ .

$$C = \{q \in \mathbb{R}^2 \mid \text{dist}(p, q) = R\}$$

We can use the distance function on the plane to come up with an implicit formula for the circle:

**Theorem 4.5** (Implicit Circle in  $\mathbb{R}^2$ ). *The circle of radius  $R$  centered at  $p = (p_x, p_y)$  is given by the implicit equation*

$$(x - p_x)^2 + (y - p_y)^2 = R^2$$

Can we also find a *parametric* description of the circle? Using the trigonometric identity  $\cos^2 \theta + \sin^2 \theta = 1$ , we see that if  $x = \cos \theta$  and  $y = \sin \theta$  then  $(x, y)$  must lie on the unit circle about the origin!

**Theorem 4.6** (Parametric Unit Circle). *The unit circle centered at  $(0, 0)$  can be parameterized as*

$$\vec{C}(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

for  $t \in [0, 2\pi]$ .

#### 4. Shapes

Translating and scaling this:

**Theorem 4.7** (Parametric Circle in  $\mathbb{R}^2$ ). *The circle of radius  $R$  centered at  $p = (p_x, p_y)$  is given by the parametric equation*

$$RC(t) + p = \begin{pmatrix} R \cos t \\ R \sin t \end{pmatrix} + \begin{pmatrix} p_x \\ p_y \end{pmatrix}$$

**Definition 4.7** (Sphere). The sphere  $S$  of radius  $R$  centered at  $p \in \mathbb{R}^3$  is the set of points in 3-dimensional space that lie at distance  $R$  from  $p$ :

$$S = \{q \in \mathbb{R}^3 \mid \text{dist}(p, q) = R\}$$

**Theorem 4.8** (Implicit Sphere in  $\mathbb{R}^3$ ). *The sphere of radius  $R$  centered at  $p = (p_x, p_y, p_z)$  is given by the implicit equation*

$$(x - p_x)^2 + (y - p_y)^2 + (z - p_z)^2 = R^2$$

It is also possible to make a parametric equation of a sphere. Since the surface of a sphere is two dimensional we will need *two parameters* much like we did for planes. The expression looks a bit complicated the first time you see it, and while we will not need it until later in the course, it appears below for completeness.

**Theorem 4.9** (Parametric Unit Sphere). *The sphere of radius 1 centered at  $(0, 0, 0)$  can be parameterized by*

$$S(u, v) = \begin{pmatrix} \cos(u) \sin(v) \\ \sin(u) \sin(v) \\ \cos(v) \end{pmatrix}$$

For  $u \in [0, 2\pi]$  and  $v \in [0, \pi]$ .

Below is a program illustrating the parametric unit sphere: the rectangle on the bottom is the space of *parameters* (the  $u - v$  coordinates), and you can track the red, blue lines and the black point between the inputs in the rectangle and the outputs in the sphere.

Try going into the menu and messing with the sliders under *domain*, to see what the two different angles do.

<https://stevejtrettel.site/code/2023/tangent-plane>

Just as for the circle, we can take this parameterization for the unit sphere and use it to find one for *any sphere* by scaling and translating it:

**Theorem 4.10** (Parametric Sphere). *The sphere of radius  $R$  centered at  $p = (p_x, p_y, p_z)$  is given by*

$$RS(u, v) + p = R \begin{pmatrix} \cos(u) \sin(v) \\ \sin(u) \sin(v) \\ \cos(v) \end{pmatrix} + \begin{pmatrix} p_x \\ p_y \\ p_z \end{pmatrix}$$

## 4.4. Other Shapes

The equation  $x^2 + y^2 = 1$  is the implicit equation for a circle in the plane. But what shape does this determine in three dimensions?

**Example 4.1** (Cylinder). A *cylinder* is a set of points in  $\mathbb{R}^3$  which project orthogonally onto a circle in some plane. That is, a “stack of circles” along some axis.

The easiest examples are cylinders around *coordinate axes*: For example,  $x^2 + y^2 = 1$  is a **circle** in  $\mathbb{R}^2$  but is a **cylinder** in  $\mathbb{R}^3$ , as the  $x, y$  coordinates make a circle but the  $z$  coordinate is free to be anything.

Similarly,  $y^2 + z^2 = 4$  is a circle of radius 2 around the  $x$ -axis.

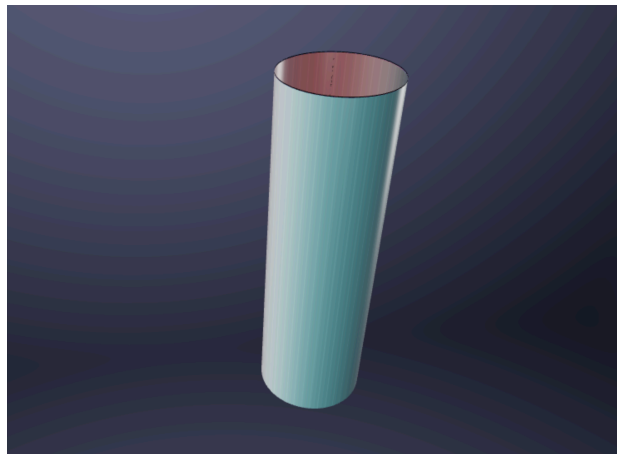


Figure 4.1.: The cylinder  $x^2 + y^2 = 1$  in  $\mathbb{R}^3$ .

In general, the implicit equation of any 2-d shape can be used in 3 dimensions to describe a “stack” of those two dimensional shapes along the direction of the variable that’s missing from the formula.

#### 4. Shapes

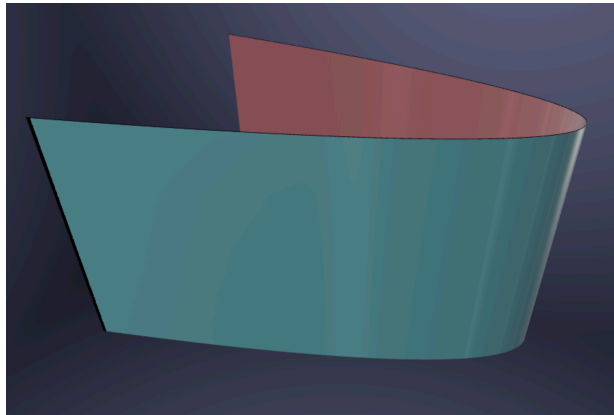


Figure 4.2.: The parabolic cylinder  $y = x^2$  in  $\mathbb{R}^3$ .

**Example 4.2.** The equation  $y = x^2$  traces out a **Parabolic cylinder**, that is, a stack of parabolas in the  $z$ -direction.

Other shapes that will be useful are ellipsoids, which are squashed spheres:

**Example 4.3** (Ellipsoid). An ellipsoid is given by the implicit equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

This is similar to the equation of an ellipse but with one more variable.

**Example 4.4** (Paraboloid). A paraboloid is the surface  $z = x^2 + y^2$ , or a stretched version  $z = ax^2 + by^2$  where  $a$  and  $b$  are positive. Its cross sections are circles (in the first case) and ellipses (in the second).

**Example 4.5** (Saddle). A saddle shaped surface, or *hyperbolic paraboloid* is a surface of the form  $z = x^2 - y^2$  or a scaling of it. In this form, it has a cross section like a upwards-facing parabola along the  $x$  axis, and a downwards-facing parabola along the  $y$  axis.

4.4. Other Shapes

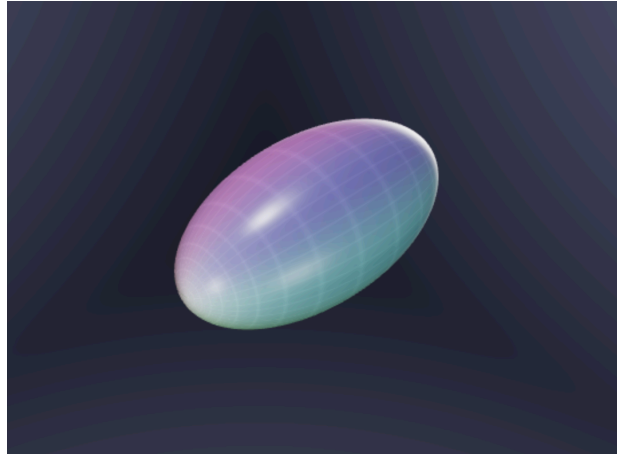


Figure 4.3.: The ellipsoid  $\frac{x^2}{2} + y^2 + z^2 = 1$  in  $\mathbb{R}^3$ .

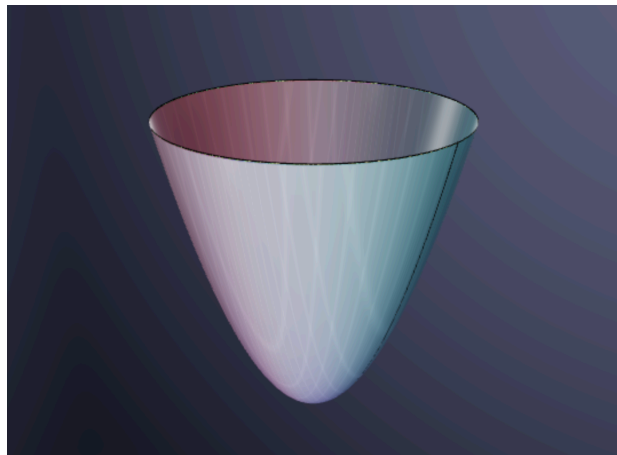


Figure 4.4.: The paraboloid  $z = x^2 + y^2$  in  $\mathbb{R}^3$ .

#### 4. Shapes

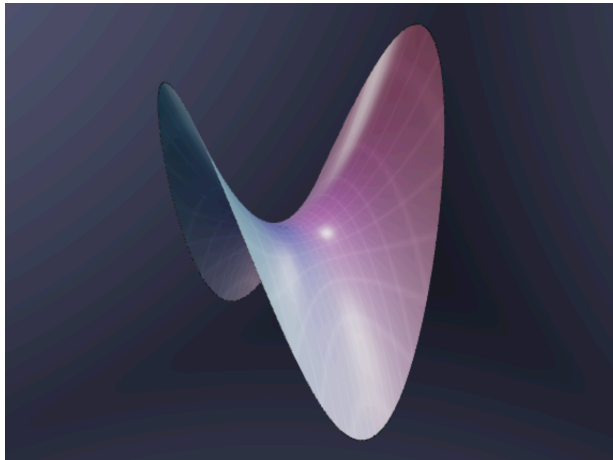


Figure 4.5.: The hyperbolic paraboloid  $z = x^2 - y^2$  in  $\mathbb{R}^3$ .

##### 4.4.1. Helpful Videos

A discussion of implicit vs parametric equations (the Calculus Blue series)

<https://youtu.be/UuYXPaac7gU?si=f8AQSDcnkNORl8-1>

Here's some videos reviewing some of the techniques we learned in class: first, a couple involving planes.

<https://youtu.be/2sZKZHyaQJ8?si=Ld79HXRv12YfQaL2>

<https://youtu.be/rL9UXzZYYo4?si=6IwekLHYirpgr1Y>

And secondly, a review of precalculus material on putting circle equations into standard form.

[https://youtu.be/u\\_39J-syjB0?si=rJYGjMXPwy8W5eYe](https://youtu.be/u_39J-syjB0?si=rJYGjMXPwy8W5eYe)

Examples of quadratic surfaces

<https://youtu.be/5y1bhGsYG8o?si=aYYIh8kOkN6b-KPn>

**Part II.**

# **Curves**





## 5. Parameterization

(Relevant Section of the Textbook: 13.1 Vector Functions and Space Curves)

**Definition 5.1** (Plane Curve). A plane curve is a function  $\vec{c}: \mathbb{R} \rightarrow \mathbb{R}^2$ . Written in the coordinates  $(x, y)$  of  $\mathbb{R}^2$ , a plane curve can be expressed using two *coordinate functions*

$$\vec{c}(t) = (x(t), y(t))$$

We've seen examples of plane curves already, for instance parametric lines (like  $\ell(t) = (2t - 1, 3t + 4)$ ) and parametric circles, (like  $c(t) = (2 \cos(t) + 1, 2 \sin(t) - 1)$ ).

**Definition 5.2** (Space Curve). A space curve is a function  $\vec{r}: \mathbb{R} \rightarrow \mathbb{R}^3$ . Written in the coordinates  $(x, y, z)$  of  $\mathbb{R}^3$ , a space curve can be expressed using three *coordinate functions*

$$\vec{r}(t) = (x(t), y(t), z(t))$$

<https://stevejtrettel.site/code/2023/parametric-curve-animation>

We see curves like this in everyday life - watching a bird fly through the air we see its position changing in time, so its  $x$ ,  $y$ , and  $z$  components are all changing in time  $(x(t), y(t), z(t))$ . Likewise, watching your car driving on a GPS, we see the car's position changing - its latitude and longitude are functions of time  $\text{car}(t) = (\text{lat}(t), \text{long}(t))$ . Indeed this is how we usually will use plane and space curves to trace out *positions* as a function of time. But it is also often useful to think of a curve as being traced out by a little arrow based at the origin (the vector picture, vs the position picture). This is particularly helpful when trying to *build* curves for yourself, as you can think about adding on terms, scalar multiplication, etc.

Here's a graphing calculator for curves, so you can try making some of your own. Can you make this draw a circle in the  $yz$  plane?

<https://stevejtrettel.site/code/2023/parametric-curve>

Parametric curves are used in animation, physics, and engineering. In this visualization below, each little blob is animated using a parametric curve: this could trace out a school of fish swarming at sea, for instance.

<https://stevejtrettel.site/code/2022/integral-curves>

## 5. Parameterization

### 5.1. Parameterization Tips

Studying the *properties* of parametric curves falls squarely within mathematics, and we will spend much time soon developing the calculus to do so. But *creating* parametric curves is more art than science: it really helps to build up some intuition for a few basic examples, and then learn how to combine them and modify them to produce new and more interesting curves. I encourage you to follow along in the discussions below using the graphing calculator at the top of the page.

**Example 5.1** (Parametrizing The Graph of a Function). If  $y = f(x)$  is a function, its graph consists of the  $y$  value  $f(x)$  whenever the  $x$ -value is  $x$ . That means, the graph of  $f$  consists of the points  $(x, f(x))$  in the plane. Expressed yet a third way, a parametric equation that traces out the graph is given by

$$\vec{c}(t) = (t, f(t))$$

Recall that an *implicit* equation gives a relationship of  $y$  and  $x$  that are satisfied by an equation. Some of these express functions like  $y = x^2$ , but others do not, for instance  $x^2 + y^2 = 1$ . Oftentimes given an implicit equation its desirable to *parameterize it*: to find a way to trace out the curve as a function of  $t$ . There's no single way to do this, and it often takes some trial and error. But some tips are below.

**Example 5.2** (Parametrizing circles). The implicit equation for the unit circle is  $x^2 + y^2 = 1$ . Because the functions  $\cos t$  and  $\sin t$  satisfy the equation

$$\cos^2(t) + \sin^2(t) = 1$$

We see that if  $x = \cos(t)$  and  $y = \sin(t)$  then  $(x, y)$  must lie on the unit circle. Similarly, the equation  $f(t) = (r \cos(t), r \sin(t))$  parameterizes a circle of radius  $r$  centered at  $(0, 0)$ , and

$$f(t) = (r \cos(t) + h, r \sin(t) + k)$$

parameterizes a circle of radius  $r$  centered at  $(h, k)$ .

Similar parameterizing an implicit plane curve by finding functions which satisfy the relations, one can parameterize curves that are the intersection of two known surfaces.

**Example 5.3.** Parameterize the twisted cubic, lying on  $y = x^2$  and  $z = x^3$ . Here if  $x = t$  then we know  $t = t^2$  and  $z = t^3$ : this fully specifies a point in 3-dimensional space, so we have our parameterization

$$f(t) = (t, t^2, t^3)$$

## 5.1. Parameterization Tips

**Example 5.4.** Parameterize the intersection of the cylinder  $x^2 + y^2 = 3$  and the plane  $x + y + z = 1$ . On the cylinder,  $x$  and  $y$  both lie on a circle of radius  $\sqrt{3}$ : so we can write  $x = \sqrt{3} \cos(t)$  and  $y = \sqrt{3} \sin(t)$ . The cylinder equation doesn't tell us anything about  $z$ , so its no help there. But - we can solve the plane for  $z$  in terms of  $x$  and  $y$  to get  $z = 1 - x - y$ . Now we can plug in what we know  $x$  and  $y$  to be to get the parameterization:

$$\gamma(t) = (\sqrt{3} \cos(t), \sqrt{3} \sin(t), 1 - \sqrt{3} \cos(t), \sqrt{3} \sin(t))$$

**Example 5.5.** Intersection of  $4y = x^2 + z^2$  and  $y = x$ . We know already because  $y = x$  that our points in space are going to look like  $(x, x, z)$ . We can substitute this idea into the first equation to see that  $4y$  becoes  $4x$ , and so

$$x^2 - 4x + z^2 = 0$$

This is the equation for a circle! We can find its center and radius by completing the square:

$$x^2 - 4x = x^2 - 4x + 4 - 4 = (x - 2x)^2 - 4$$

So, this is the circle

$$(x - 2)^2 + z^2 = 4$$

Which is a circle of radius 2 centered at  $(2, 0)$ . We can parameterize it as  $x = 2 \cos(t) + 2$  and  $z = 2 \sin(t)$ . So, in 3D along the plane  $y = x$  this becomes

$$f(t) = (2 \cos(t) + 2, 2 \cos(t) + 2, 2 \sin(t))$$

### 5.1.1. Same Curve, Different Parameterizations

Different parameterizations can describe the same curve: since a parameterization is like an *animation* of the curve, you can think of this as tracing out the curve at different speeds.

**Example 5.6** (Different Parameterizations of the Circle). All three of these parametric curves trace out the unit circle.

$$(\cos t, \sin t)$$

$$(\cos 2t, \sin 2t)$$

$$(\cos t, -\sin t)$$

The first traces it at unit speed, counterclockwise. The second at twice the speed in the same direction. And the third, at unit speed but *backwards* (clockwise).

## 5. Parameterization

**Example 5.7** (Different Parameterizations of  $y^2 = x^3$ ). We can parameterize the implicit curve  $y^2 = x^3$  in several ways: taking the square root of both sides gives  $y$  as a function of  $x$  (with a plus and minus component),  $y = \pm\sqrt{x^3}$  so one possible parameterization is

$$\alpha(t) = (t, \pm\sqrt{t^3})$$

This isn't the nicest, as we have that  $\pm$  sign. Another thing we could do is take the cube root: this doesn't cause any  $\pm$  ambiguity, and gives  $x$  as a function of  $y$ , or  $x = \sqrt[3]{y^2}$ , leading to the parametric curve

$$\beta(t) = (\sqrt[3]{t^2}, t)$$

A third option is to find a function for  $x(t)$  where when we cube it, we get the same thing as if we squared the function we chose for  $y(t)$ . This is of course trickier - but here one option is to take  $x = t^2$  and  $y = t^3$ . Then  $x^3 = t^6$  and  $y^2 = t^6$  so  $x^3 = y^2$  and our curve is

$$\gamma(t) = (t^2, t^3)$$

### 5.1.2. New Curves from Old

Once we know a few parametric curves (circles, lines, some implicit curves, etc) - its easy to find more by *modifying* the ones we already know! Some of the simplest such transformations we've already used in the case of circles, scaling and translation.

**Theorem 5.1** (Scaling a Parametric Curve). *If  $f(t) = (x(t), y(t))$  is a parametric curve, then  $rf(t) = (rx(t), ry(t))$  is a curve where all the coordinates are  $r$  times a big.*

**Theorem 5.2** (Translating a Parametric Curve). *If  $f(t) = (x(t), y(t))$  is a parametric curve, then  $f(t) + (a, b) = (x(t) + a, y(t) + b)$  is the result of shifting the curve over by  $(a, b)$ .*

Of course, more interesting transformations are also possible - and it's easiest to see this through a couple examples!

## 5.2. Case Study: Spirals

We will make and understand a collection of *spirals* starting with the basic equation of the unit circle

$$(\cos(t), \sin(t))$$

**Example 5.8** (Archimedean Spiral). The archimedean spiral is the curve that rotates about the origin at unit speed, but after rotating angle  $t$ , lies not at unit distance (like a circle) but at distance  $t$  from the origin. To parameterize, we *multiply* the circle by  $t$ :

$$\gamma(t) = (t \cos(t), t \sin(t))$$

**Example 5.9** (Logarithmic Spiral). The logarithmic spiral moves away from the origin *exponentially fast*, instead of linearly. This has radius at time  $t$  equal to  $e^t$ , so

$$\gamma(t) = (e^t \cos(t), e^t \sin(t))$$

Different functions  $r(t)$  for the radius multiplied by the circle give spirals that move outwards (or inwards) at different speeds. Try making some of these in the graphing calculator above!

**Exercise 5.1** (Whirlpool). Can you make a spiral that rotates about the origin at unit speed, but whose radius asymptotes to 2, never getting any larger?

**Example 5.10** (Helix). A helix is a curve where  $x, y$  travel around a circle, and  $z$  increases at unit speed. For example, the unit helix is

$$\gamma(t) = (\cos(t), \sin(t), t)$$

**Example 5.11** (Slinky-Like Helix). What if we want a helix like curve to move vertically at an uneven rate? Replace the  $z$  component with a more interesting function of  $t$ ! For instance, if  $z = e^t$  then the curve bunches up as  $t \rightarrow -\infty$  along the  $xy$  plane:

$$\gamma(t) = (\cos(t), \sin(t), e^t)$$

**Example 5.12** (Spiral On a Cone). The surface  $z = \sqrt{x^2 + y^2}$  traces out a cone - the height is equal to the radius! How can we draw a spiral on the surface of the cone? Well, if we know what we want the spiral to do in its  $x$  and  $y$  components, we can calculate the  $z$  component using the formula above! For instance, given the archimedean spiral  $(t \cos(t), t \sin(t))$  we see  $z = \sqrt{(t \cos t)^2 + (t \sin t)^2} = t$ . Thus, the curve is

$$\gamma(t) = (t \cos t, t \sin t, t)$$

## 5. *Parameterization*

### **5.3. Videos**

A recap of parametric curves

<https://youtu.be/bb4bSCjFAw?si=3cHJuYIJvD9N6VuD>

Parametric curves and elimination of parameters:

<https://youtu.be/97pe-QISGqA?si=pKlz2u9f3an3fPC3>

## 6. Calculus

**(Relevant Section of the Textbook: 13.2 Derivatives and Integrals of Vector Functions)**

A parametric curve is made of  $n$  component functions, which are familiar functions  $\mathbb{R} \rightarrow \mathbb{R}$  from single variable calculus. Thus, the calculus of curves is simply doing the calculus of single variable functions  $n$  times!

### 6.1. Limits

Taking limits of a parametric curve is just taking the limit of each component function.

**Definition 6.1** (Limits of Parametric Curves). If  $\vec{r}(t) = (x(t), y(t), z(t))$  is a parametric curve, then limits are computed componentwise:

$$\lim_{t \rightarrow a} r(t) = \left( \lim_{t \rightarrow a} x(t), \lim_{t \rightarrow a} y(t), \lim_{t \rightarrow a} z(t) \right)$$

**Example 6.1** (Limits of Parametric Curves). Let  $\vec{r}(t)$  be the following parametric curve

$$\vec{r}(t) = \left\langle \frac{1}{t+1}, \frac{\sin(t)}{t}, \frac{3t+t^2}{t} \right\rangle$$

Compute the limit  $\lim_{t \rightarrow 0} \vec{r}(t)$ .

Computing the limit componentwise we see we just need to evaluate three limits:

$$\begin{aligned} \lim_{t \rightarrow 0} \frac{1}{t+1} \\ \lim_{t \rightarrow 0} \frac{\sin(t)}{t} \\ \lim_{t \rightarrow 0} \frac{3t+t^2}{t} \end{aligned}$$

## 6. Calculus

The first of these is continuous at zero so we can just plug in. The second need L'Hospital's rule, and the third needs us to cancel a  $t$  from the numerator and denominator before plugging in, to get

$$\lim_{t \rightarrow 0} \vec{r}(t) = \langle 1, 1, 3 \rangle$$

### 6.2. Differentiation

Recall the single variable definition of the derivative:

$$f'(t) = \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}$$

The same definition works for parametric curves:

**Definition 6.2** (Differentiating Curves). The derivative of a parametric curve  $\vec{r}(t)$  at a point  $t$  is given by the following limit:

$$\lim_{h \rightarrow 0} \frac{\vec{r}(t+h) - \vec{r}(t)}{h}$$

The numerator here is a *vector*, which gets infinitesimally small as  $h \rightarrow 0$ , connecting two closer and closer together points of the curve. We then rescale this vector with scalar multiplication, dividing by  $h$  to keep the vector's length from collapsing. In the limit, this converges to a vector  $\vec{r}'(t)$  which is *tangent* to the curve.

<https://stevejtrettel.site/code/2023/parametric-curve-tangent>

However, we don't need to calculate the derivative using this limit statement every time! We can use the fact that limits distribute over the components of the function to prove that we can also take the derivative *one component at a time*.

**Theorem 6.1** (Differentiating Curves Componentwise). If  $\vec{r}(t) = (x(t), y(t), z(t))$  is a parametric curve, then

$$\vec{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$$

Because of this, it's straightforward to show that differentiation of vector functions obeys the familiar laws of single variable calculus: you can break it up over sums, and pull out scalars. But now there are three types of products (do we multiply the vector function by a scalar function, or dot or cross product it with another vector?)



**Theorem 6.2** (Differentiation Product Laws).

$$\begin{aligned}(f(t)\vec{r}(t))' &= f'(t)\vec{r}(t) + f(t)\vec{r}'(t) \\ (\vec{c}(t) \cdot \vec{r}(t))' &= \vec{c}'(t) \cdot \vec{r}(t) + \vec{c}(t) \cdot \vec{r}'(t) \\ (\vec{c}(t) \times \vec{r}(t))' &= \vec{c}'(t) \times \vec{r}(t) + \vec{c}(t) \times \vec{r}'(t)\end{aligned}$$

There is also a chain rule: we can't compose a vector function *inside* another vector function, but we can plug a scalar function in as the parameter in a curve!

**Theorem 6.3** (The chain rule).

$$(\vec{r}(f(t)))' = \vec{r}'(f(t))f'(t)$$

You probably notice a similarity to the single variable calculus versions in all of these: they're as close as possible, except now being about vector functions! But these simple looking rules actually provide us a new powerful set of tools, they tell us about the *rate of change* at the same time as dot and cross products - which we know can measure areas and angles!

As one quick example, we'll prove a very useful fact about curves defined by vectors of constant length.

**Theorem 6.4** (Curves on the Sphere). *If a curve  $\vec{r}(t)$  never changes in length, so  $\|\vec{r}(t)\| = k$  for all time, then  $\vec{r}'$  is orthogonal to  $\vec{r}$ .*

*Proof.* Since the magnitude of a vector can be calculated from its dot product, we see that

$$\vec{r} \cdot \vec{r} = k^2$$

Taking the derivative of this with the product rule, we find

$$\vec{r}' \cdot \vec{r} + \vec{r} \cdot \vec{r}' = 0$$

The dot product is commutative (order doesn't matter) so we can re-arrange the left hand side:

$$2\vec{r} \cdot \vec{r}' = 0$$

But dividing by two - this says that the dot product of  $\vec{r}$  and  $\vec{r}'$  is zero! So these two vectors are orthogonal.

□

### 6.3. Integration

The story of vector valued integration is similarly straightforward. Recall the definition of integration via Riemann sum:

$$\int_a^b f(x)dx = \lim_{N \rightarrow \infty} \sum_{i=1}^N f(x_i)\Delta x$$

We can apply the same definition to vector valued functions, as its composed of just the operations of addition and scalar multiplication!

**Definition 6.3** (Integrating Curves). If  $\vec{r}$  is a parametric curve, its *vector valued integral* on the interval  $[a, b]$  is defined by the following Riemann sum:

$$\int_a^b \vec{r}(t)dt = \lim_{N \rightarrow \infty} \sum_{i=1}^N \vec{r}(t_i)\Delta t$$

But wait! Both scalar multiplication and addition are things we can do *componentwise* for a vector. So we can break this big Riemann sum up into a standard riemann sum in each component. Taking the limit, this tells us we can integrate vector functions componentwise.

**Theorem 6.5** (Integrating Curves Componentwise). If  $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$ , the *vector valued integral of  $\vec{r}$  on  $[a, b]$*  is given by

$$\int_a^b \vec{r}(t)dt = \left\langle \int_a^b x(t)dt, \int_a^b y(t)dt, \int_a^b z(t)dt \right\rangle$$

We will only find limited use of this in our class, as there are other types of integrals along curves that will prove more important. Nonetheless this does show up in many applications of multivariable calculus to engineering and physics, where one may wish to recover position from velocity, or velocity from acceleration.

**Example 6.2** (Displacement from Velocity). If  $\vec{v}(t)$  is a parametric curve giving the *velocity* of a particle at time  $t$ , then  $\int_a^b \vec{v}(t)dt$  is teh *displacement vector* for its net travel between  $t = a$  and  $t = b$ .

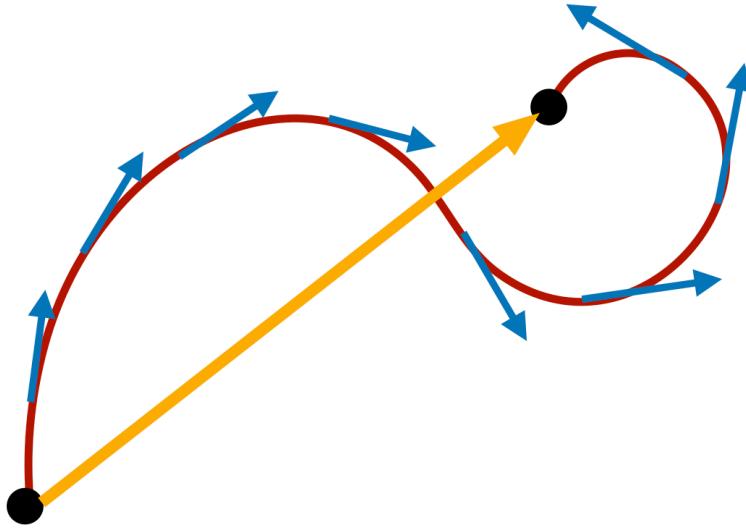


Figure 6.1.: The displacement (yellow vector) can be calculated by integrating the velocity vectors (blue) even if we do not know the parametric curve itself (red).

The same is true for *acceleration*: all of our phones have a sensor inside called a three axis accelerometer. These measure the acceleration as a function of time

$$\vec{a}(t) = \langle a_x(t), a_y(t), a_z(t) \rangle$$

But this is not what software running on the phone wants or needs: it cares about your *position* in space! (Say, if you're using your phone in augmented reality). To get this, it *integrates* the acceleration to get velocity, and then *integrates again* the velocity to get position! P For any engineers in the class - if you have accelerometers in the Hive: this could be a fun project to code up! Write a small python program to numerically integrate (ie compute a Riemann sum) from the output data of an accelerometer, and track your hand's position as you move it around.

## 6.4. Videos

Calculus Blue series on calculus of curves:

<https://youtu.be/8WzHSgE0Kus?si=D217ouCNQBLxRsI9>

Here are some videos practicing the concepts that we have done examples of in lecture. First, limits of vector functions.

## 6. Calculus

<https://youtu.be/bhP9cfB90Kc?si=pS7CeIkhktSTijRu>

Next, derivatives of vector functions.

<https://youtu.be/i9FugTcqWKO?si=p7BoWwVic8-6d-Wu>

<https://youtu.be/vcwvzUVLPw0?si=mtZpSwCJVJ5Gu5gi>

And the proofs of the differentiation laws for the dot product and cross product:

[https://youtu.be/fbzEYaYogfo?si=9aw3lktC0ciLq\\_Dp](https://youtu.be/fbzEYaYogfo?si=9aw3lktC0ciLq_Dp)

<https://youtu.be/vykDXI9OjDM?si=3zDR8TnwSPPSsP9>

Also, integrating vector functions.

<https://youtu.be/RsGuE5OZqMg?si=onfxxJTYrVZ5dTu>

## 7. Geometry

(Relevant Section of the Textbook: 13.3 Arc Length and Curvature)

### 7.1. Arc Length

**Definition 7.1** (Infinitesimal Arclength). If  $\vec{r}(t)$  is a parametric curve, its infinitesimal arclength is measured by

$$ds = |\vec{r}'(t)| dt$$

This makes sense: after all the derivative  $\vec{r}'(t)$  is the *velocity*,  $\|\vec{r}'(t)\|$  is the *speed*, and  $dt$  is an infinitesimal length of time. Thus, the product  $\|\vec{r}'(t)\|dt$  is an infinitesimal bit of *distance* - a small length along the curve. To take this infinitesimal information and get something useful out - we need to *integrate* along the curve.

**Definition 7.2** (Arclength). If  $\vec{r}(t)$  is a parametric curve, its length between  $t = a$  and  $t = b$  is given by

$$L = \int_a^b ds = \int_a^b |\vec{r}'(t)| dt$$

**Example 7.1** (Arclength of a Helix). Find the arclength of  $\vec{r}(t) = (\cos(t), \sin(t), t)$  from  $t = 0$  to  $t = 2\pi$ . First, we need to find the velocity  $\vec{r}'$ :

$$\vec{r}'(t) = \langle -\sin(t), \cos(t), 1 \rangle$$

Next, we need to take this velocity and find the speed:

$$\|\vec{r}'(t)\| = \sqrt{(-\sin t)^2 + (\cos t)^2 + 1} = \sqrt{2}$$

Finding arclength is just integrating this over the domain:

$$\int_0^{2\pi} \|\text{vecr}'(t)\| dt = \int_0^{2\pi} \sqrt{2} dt = 2\pi\sqrt{2}$$

## 7. Geometry

Often arclength integrals can be challenging to do, because of the square root. But with some algebra and integration tricks, a lot can be learned.

From this idea, we can define the *arclength function* which measures the length of a curve  $\vec{r}(t)$  from a starting point  $t = a$ :

**Definition 7.3** (The Arclength Function). If  $\vec{r}(t)$  is a parametric curve, for any given starting point  $t = a$  we may define the *arclength function* which measures the length of curve between  $a$  and  $t$ :

$$s(t) = \int_a^t |\vec{r}'(u)| du$$

(Note we have changed the variable of integration so that  $t$  is not used in two different contexts)

What is the arclength function for the helix in our earlier example, starting from  $t = 0$ ? Since  $\|\vec{r}'(t)\| = \sqrt{2}$ , we see that

$$s(t) = \int_0^t \sqrt{2} dt = \sqrt{2}t$$

This tells us that after  $t$  seconds, we have traced out  $\sqrt{2}t$  units of arclength. How could we reparameterize this curve so that its arclength function is just  $s(t) = t$  (tracing out  $t$  units of arc in  $t$  units of time)?

**Definition 7.4** (Unit Speed Curve). A curve  $\vec{c}(t)$  is *unit speed* if  $\|\vec{c}'(t)\| = 1$  for all times  $t$ . This means that it after  $t$  seconds, the curve has traversed  $t$  units of length. For this reason, we also call unit speed curves *arclength parameterized curves*.

In our example, to make the helix unit speed we need to slow it down by a factor of  $\sqrt{2}$ : that is, we need  $\vec{r}(t/\sqrt{2})$ :

$$\vec{r}\left(\frac{t}{\sqrt{2}}\right) = \left(\cos \frac{t}{\sqrt{2}}, \sin \frac{t}{\sqrt{2}}, \frac{t}{\sqrt{2}}\right)$$

## 7.2. Curvature

Besides the length of a curve, one of the most powerful things calculus allows us to do is rigorously study its *curvature*. How can we quantify the fact that some curves bend gently and others turn sharply in space? One means of trying to do this is by looking at the tangent vectors to the curve, and trying to determine how quickly they are changing.

Of course, there's a complication to this: a tangent vector can change in *length* without changing in direction. This doesn't mean that a curve is *curving*, but rather that the particle tracing it out is *accelerating*.

To remove this worry, we define the *unit tangent vector* to a curve. Just divide the derivative by its magnitude!

**Definition 7.5** (Unit Tangent Vector). The unit tangent vector to the curve  $\vec{r}(t)$  is the vector of length 1 which is parallel to  $\vec{r}'(t)$ :

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

<https://stevejtrettel.site/code/2023/parametric-curve-tangent>

This allows a clean definition of curvature: it is how much the unit tangent vector turns per arclength.

**Definition 7.6** (Curvature of a Curve). The curvature of a curve is

$$\kappa = \left| \frac{d\vec{T}}{ds} \right| = |\vec{T}'(t)| / |\vec{r}'(t)|$$

Where the second equality is derived via the chain rule:

$$\left| \frac{d\vec{T}}{ds} \right| = \left| \frac{d\vec{T}}{dt} \frac{dt}{ds} \right| = \left| \frac{d\vec{T}}{dt} \right| \left| \frac{dt}{ds} \right| = \frac{|\vec{T}'|}{|\vec{r}'|}$$

This formula is difficult to apply in general, as the unit tangent vector  $T$  might have a pretty scary looking formula, and so taking its derivative can be a lot of work.

Doing some calculus we can get a simpler formula:

## 7. Geometry

**Theorem 7.1** (Curvature of a Curve). *The curvature of  $\vec{r}(t)$  is given by*

$$\kappa(t) = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

This is something that's relatively easy to compute (though perhaps tedious) from any parameterization: you just need to find the first and second derivatives, take a cross product, and then plug into the formula!

But, if we further restrict ourselves to the case that  $\vec{r}(t) = (t, f(t))$  traces the graph of a function, we can simplify this calculation even more:

**Theorem 7.2** (Curvature of a Graph). *If  $y = f(x)$  is a function, the curvature of its graph is*

$$\kappa(x) = \frac{|f''(x)|}{|1 + f'(x)|^{3/2}}$$

The below graphing calculator lets you enter a function  $\kappa(s)$  that specifies the curvature of a curve, and then it computes a curve which has that curvature! Try even just the case  $k(s) = s$  and think about the result - what sort of curve do you expect to see if the curvature grows linearly along the length of the curve?

<https://stevejtrettel.site/code/2022/curvature-torsion>

### 7.3. Framing a Curve

The unit tangent vector provides us with a very useful “pointer” - always oriented directly along a curve. But in any serious application of parametric curves, we need more information: we would like a whole  $x, y, z$  coordinate frame at each point of the curve.

To start, we'll look for one vector which is orthogonal to, or *normal* to our curve. How can we find one? Well, the unit tangent is of constant length (its the *unit tangent*, after all). We can use the *product rule for dot products* to understand  $T'$ :

$$(T \cdot T) = 1 \implies (T \cdot T)' = 0$$

$$(T \cdot T)' = T' \cdot T + T \cdot T' = 2T \cdot T'$$

Thus, we see that  $2T \cdot T' = 0$ , so  $T$  is orthogonal to  $T'$ !. To find a unit vector orthogonal to  $T$ , we just need to normalize  $T'$ .



**Definition 7.7** (Normal Vector).

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

Given these two, its easy to find a third unit vector: just take the cross product of  $T$  and  $N$ ! The result is called the *binormal* as its a second normal vector to the curve.

**Definition 7.8** (Binormal Vector).

$$\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

Together these three vectors provide a coordinate system at each point along the curve:  $T$  measures distance in the tangent direction to the curve,  $N$  measures distance in the direction the curve is bending fastest, and  $B$  is orthogonal to both. This collection of vectors is called the *Frenet Frame* and is heavily used in computations in physics, engineering, and computer graphics.

**Definition 7.9** (Frenet Frame).

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)} \quad \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)} \quad \vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$$

<https://stevejtrettel.site/code/2023/frenet-frame>

## 7.4. Videos

Here are some useful videos reviewing the sort of examples that we have covered in class:

### 7.4.0.1. Arclength of Curves

<https://youtu.be/TAQPEP9pEhw?si=EIy61FdfPevHZnR0>

[https://youtu.be/TZJ2btou8\\_c?si=XjVF1w0CfbcgxXKC](https://youtu.be/TZJ2btou8_c?si=XjVF1w0CfbcgxXKC)

<https://youtu.be/X3AAltBz0wo?si=mcKCWeQMpjz5aO0>

Parameterizing a curve with respect to arclength (a unit speed parameterization)

<https://www.youtube.com/watch?v=O3nnibgLCCc>

## 7. Geometry

### 7.4.0.2. Unit Tangents and Normals

[https://youtu.be/mDUpR1\\_Qn70?si=RSIHcXX9-1eKMOOr](https://youtu.be/mDUpR1_Qn70?si=RSIHcXX9-1eKMOOr)

The calculus Blue series on Tangent, Normal and Curvature:

[https://youtu.be/qfE2nTaLxD8?si=zBJ\\_cAFGtVTYEu6z](https://youtu.be/qfE2nTaLxD8?si=zBJ_cAFGtVTYEu6z)

The binormal vector and the Frenet Frame:

[https://youtu.be/VkqTYPq8dX4?si=c0t\\_GzVKFVwF3dn8](https://youtu.be/VkqTYPq8dX4?si=c0t_GzVKFVwF3dn8)

### 7.4.0.3. Curvature

<https://youtu.be/NlcvU67YWpQ?si=ImoRnmwFO0ALQC1R>

An application of this: finding the point on a curve where it is maximally curved (say, you wanted to find the sharpest bend of a roller coaster)

[https://youtu.be/qpbFlhDbKwI?si=8IhMtdnVH7kZ\\_MAh](https://youtu.be/qpbFlhDbKwI?si=8IhMtdnVH7kZ_MAh)

## 8. Differential Equations

**(Relevant Section of the Textbook: Only a portion of 13.4: Motion in Space, Velocity and Acceleration. Not the whole section)**

One of the most striking uses for the theory of parametric curves is in describing the world around us. Indeed the original inventions of calculus, parametric curves, and modern physics all can be traced to the same source: Issac Newton and his desire to predict the future. Newton's big insight was that this lofty goal could be converted into something concrete: the study of differential equations of parametric curves.

### 8.1. Projectile Motion

If I drop an apple from my hand above the ground, what will it do next? Can I *predict* the future using some mathematical regularities of nature? Of course - I can predict it *qualitatively*- the apple will fall. But this isn't what we (or Newton) are after. We seek an *exact prediction* - of exactly where the apple will be at every point in time after we drop it.

Finding such a prediction sounds daunting indeed: but the first step towards a solution is to realize that what we are after - the exact location  $(x, y, z)$  of the apple at each time - is a parametric curve!

$$\text{apple}(t) = (x(t), y(t), z(t))$$

[The three words that I've *never* managed to spell correctly are *business*, *necessary*, and \*Gall] Thus, we seek a method of determining  $x, y$  and  $z$  as functions of  $t$ . Of course this is the hard part! We need something to grab on to: and luckily, the key insight was already discovered by Galileo nearly 500 years ago.

**Every object accelerates downwards by the same amount, regardless of what its made of or how big it is**

The story usually goes that this was discovered by dropping items from the Leaning Tower of Pisa - but Galileo actually did something much more precise, rolling objects

## 8. Differential Equations

down ramps of various slopes and measuring their positions over time consistently by counting heartbeats. In the end, we learned that the *acceleration vector* of our curve can be written

$$\vec{a} = \langle 0, 0, -g \rangle$$

Where  $g$  is a constant (the amount that gravity accelerates things: here on earth its 9.8 in metric units). It's in the  $z$  component as gravity acts vertically, and its negative because gravity accelerates *downwards*.

And it is in this humble equation that all of kinematics lies. We can fully derive the position  $p(t)$  of our apple at any future time from this, and calculus. Remember, acceleration is the second derivative of position, so really

$$\vec{a} = \langle x''(t), y''(t), z''(t) \rangle$$

Equating this to the vector  $\langle 0, 0, -g \rangle$  gives us a system of *three differential equations* one for each of our mystery functions  $x, y, z$ . And we can solve these! Starting with  $x$ , we start with  $x''(t)$ , so we integrate once to get rid of one derivative:

$$x'(t) = \int x''(t) dt = \int 0 dt = C$$

Where  $C$  is a constant of integration. But since  $x'$  measures *velocity* in the  $x$  direction, this is saying that the  $x$ -velocity is *constant in time* and is equal to  $C$ . Thus we should probably rename this constant easier to interpret: let's call it  $v_x$ . Thus we know  $x'(t) = v_x$ , and to get  $x$  we integrate once more:

$$x(t) = \int x'(t) dt = \int v_x dt = v_x t + C$$

Where again  $C$  is a constant of integration. Here we see that at time  $t = 0$  we have  $x(t) = C$ , so this constant actually tells us the *initial position* so we should rename it something more memorable too, let's go with  $x_0$ . This is the entire equation for  $x$ :

$$x(t) = v_x t + x_0$$

And since  $y$  has the same differential equation  $y''(t) = 0$ , we know it will have a similar solution

$$y(t) = v_y t + y_0$$

## 8.2. Universal Gravitation

This leaves us only to solve the  $z$ -equation  $z''(t) = -g$ . Again, we integrate twice

$$z'(t) = \int z''(t) dt = \int (-g) dt = -gt + v_z$$

Where here we've already named the constant something useful since we know that it represents the *initial velocity* in the  $z$ -direction. Integrating again,

$$z(t) = \int z'(t) dt = \int -gt + v_z dt = \frac{-1}{2}gt^2 + v_z t + z_0$$

But now we know all of  $x, y, z$  in terms of six constants: three velocities  $v_x, v_y, v_z$  specifying how fast the apple is moving when we let go at  $t = 0$ , and three positions  $x_0, y_0, z_0$  specifying where the apple is at  $t = 0$ . From here, everything is completely determined

$$\text{apple}(t) = \begin{pmatrix} v_x t + x_0 \\ v_y t + y_0 \\ \frac{-1}{2}gt^2 + v_z t + z_0 \end{pmatrix}$$

All of these formulas from early physics courses are really just the simple vector equation  $\vec{a} = \langle 0, 0, -g \rangle$  unpacked.

## 8.2. Universal Gravitation

Newton built a grand new theory of gravitation that realizes the above story as a mere approximation occurring near the Earth's surface. Newton describes all of mechanics in terms of *forces*, where we can use a force to figure out acceleration via Newton's law setting them proportional:

$$\vec{F} = m\vec{a}$$

### 8.2.1. Deriving Newton's Gravitational Law

Newton conjectured a behavior for the force law of gravity between a two masses, say of size  $m$  and  $M$  to work as follows: the force on the mass  $m$  has

- Its *direction* points from the location of mass  $m$  to the location of mass  $M$ .
- Its *magnitude* is proportional to the product of the masses  $Mm$
- Its *magnitude* is inversely proportional to the distance between the masses.

## 8. Differential Equations

Can we turn these ideas into a concrete formula using our knowledge of vectors and operations? First, let's set up our coordinates  $x, y, z$  in space. Let's put one of our masses - say  $M$  at the point  $(0, 0, 0)$  because that's easy to work with. Then our goal is to compute the force at the point  $m$ : we don't know where  $m$  is relative to  $M$ , so we can just call its location  $r = (x, y, z)$ .

The *direction* of a vector is usually specified by giving a unit vector - so we need to describe the *unit vector pointing from  $(x, y, z)$  to the origin*. We know how to find a vector from one point to another: its ending point minus starting point:

$$\text{end} - \text{start} = (0, 0, 0) - (x, y, z) = (-x, -y, -z) = -\vec{r}$$

Now to make this a unit vector, we divide by its length:

$$\|\vec{r}\| = \sqrt{x^2 + y^2 + z^2}$$

So finally we get the unit vector  $\hat{u}$  pointing from  $m$  to  $M$  is

$$\hat{u} = \frac{-\vec{r}}{\|\vec{r}\|}$$

Now we need to deal with the *magnitude*. Here Newton says that we know two things: the magnitude is proportional to the product  $Mm$ , and its *inversely proportional* to the square of the distance between the masses. How do we write down the square of the distance? Well,  $r$  is a vector reaching from  $(0, 0, 0)$  to  $(x, y, z)$  so its length is the distance between  $M$  and  $m$ : thus the distance squared is

$$\|\vec{r}\|^2 = x^2 + y^2 + z^2$$

Thus, whatever the force vector  $F$  is, we know its magnitude is proportional to  $Mm/\|\vec{r}\|^2$ : this means there's some constant  $G$  for which

$$\|\vec{F}\| = \frac{GMm}{\|\vec{r}\|^2}$$

But now we are done! We know both the magnitude and direction of  $F$ , so we can directly write down a vector  $F$  satisfying Newton's requirements:

$$F = \left( \frac{GMm}{\|\vec{r}\|^2} \right) \hat{u} = \left( \frac{GMm}{\|\vec{r}\|^2} \right) \frac{-\vec{r}}{\|\vec{r}\|}$$

## 8.2. Universal Gravitation

We can simplify this a bit by combining the three factors of  $\|r\|$  we see in the denominator, and expand it out by plugging in  $r = (x, y, z)$  if we like:

$$\begin{aligned}\vec{F} &= \frac{-GMm\vec{r}}{\|r\|^3} \\ &= \frac{-GMm}{(x^2 + y^2 + z^2)^{3/2}} \langle x, y, z \rangle\end{aligned}$$

Combining this with  $F = ma$  we see that the two small  $m$ 's cancel: this is exactly Galileo's insight that the mass of the falling body doesn't affect its motion!

$$a = \frac{-GM\vec{r}}{\|r\|^{3/2}}$$

Which, when expanded out in  $x, y, z$  gives a more complicated set of equations than we've dealt with so far:

$$\begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = \frac{-GM}{(x^2 + y^2 + z^2)^{3/2}} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

This set of equations is *much* harder to solve than our earlier case, where we approximated that the earth was so big and we were so close that we could treat the acceleration as only occurring in one direction. Here, the  $x, y,$  and  $z$  all show up in all of the equations! It was an incredible triumph of Newton to actually *solve* these, and show that planets orbit the sun on Ellipses - a fact that had been known since Kepler from observation, but had no theoretical understanding. Look in our textbook, in the last section of Chapter 13, for a rather in-depth treatment of this!

Here I do not want to try and actually solve this system, but rather showcase another technique - sometimes its possible to partially *guess a solution*, and then use an equation to fix things up until it works!

### 8.2.2. The Length of a Year

Imagining a physical system like the earth orbiting the sun - we know that our particular orbit isn't perfectly circular, but it sure is close! This might make us wonder "are there any perfectly circular orbits"? Let's put our circle in the  $xy$  plane, so  $z = 0$ . If the circle is radius  $R$ , we would be proposing an orbit of the form

## 8. Differential Equations

$$r(t) = (R \cos t, R \sin t, 0)$$

Does this solve our equation? To check, we need to plug these in for  $x$ ,  $y$ , and  $z$  to see if it comes out true. Taking the second derivative gives us the proposed left side:

$$\langle x'', y'', z'' \rangle = \langle -R \cos t, -R \sin t, 0 \rangle$$

Now computing the right hand side, we first note that  $x^2 + y^2 + z^2 = R^2$  (since the path is a circle of radius  $R$ ) This makes it easier to simplify things:

$$\frac{-GM}{R^3} \langle R \cos t, R \sin t, 0 \rangle$$

If this circle *was* a solution: these two sides should be equal! But are they? No, unfortunately not. They have the right functions, and the right sign, but one side has a jumble of constants out from that the other does not!

How can we fix this? It turns out we overlooked one thing: the *speed* of our planets orbit! We just used sine and cosine directly - which traverse the unit circle at *unit speed*. This made our orbit traced out at speed  $R$ : meaning that the farther our planet was from the sun the faster it was going! This isn't anything like what we see in the world around us (Earth takes 1 year, whereas pluto goes much slower and takes 120). So, we should try to "adjust" our solution by allowing the speed to vary. As we saw when working with helices in class, we can change the speed by multiplying the time parameter by a constant. So our next guess is

$$r(t) = (R \cos(kt), R \sin(kt), 0)$$

Re-running the same math, everything is the same except the second derivative now has a factor of  $k^2$  out from (one  $k$  pops out from each derivative, by the chain rule). Thus we have

$$k^2 \begin{pmatrix} -R \cos(kt) \\ -R \sin(kt) \\ 0 \end{pmatrix} = \frac{GM}{R^3} \begin{pmatrix} -R \cos(kt) \\ -R \sin(kt) \\ 0 \end{pmatrix}$$

These are equal precisely when the constants are equal: thus we have found

$$k^2 = \frac{GM}{R^3} \quad \implies \quad k = \sqrt{GM/R^3}$$



## 8.2. Universal Gravitation

After this adjustment, we managed to describe a precise circular orbit for a planet around a star! What can we learn from this? Well, the functions  $\cos t$  and  $\sin t$  repeat every  $2\pi$ , so the functions  $\cos(kt)$  and  $\sin(kt)$  repeat every  $2\pi/k$  time units. This means the length of a year for our planet at radius  $R$  is

$$\text{Year} = \frac{2\pi}{k} = \frac{2\pi}{\sqrt{\frac{GM}{R^3}}} = \frac{2\pi}{\sqrt{GM}} R^{3/2}$$

That is, for a circular orbit there's a relationship between the year length, the radius of the orbit, and the mass of the planet. And since realistic orbits are near-ish to circles, this should approximately hold for other orbits (in fact, doing more math we figure out it *exactly* holds, if you replace the radius with the longer axis of an ellipse).

Why is this important? Well - year length and radius are two things that we can measure from earth with telescopes: but masses of objects that are light years away are much harder to pin down. And this relationship gives us a way to find the mass without ever leaving our home!

$$M = \frac{4\pi^2}{G} \frac{R^3}{\text{Year}}$$

Astrophysicists can then take this information to learn quite a lot: once you get the star's mass, you can use this plus one other measurement (the radial velocity, or slow movement of the star) to find the planet's mass itself!

<https://youtu.be/GidwIghFbYk?si=s0S09ouFzPOSzDsr>

### 8.2.3. Living in 3-dimensions

(You are not responsible for this, so I will not write much detail here: but I may discuss it a bit in class, if I have time).

<https://stevejtrettel.site/code/2021/gravity-power-law/>

### 8.2.4. The 3-body Problem

(You are not responsible for this, so I will not write much detail here: but I may discuss it a bit in class, if I have time).

<https://stevejtrettel.site/code/2022/three-body/>

### 8.3. Electromagnetism

The force of gravity isn't the only thing that can be understood by Newton's paradigm of forces and accelerations. In the 1800s a mathematical understanding of electricity and magnetism began to come together. We learned that in addition to mass, certain particles have a *charge*, which lets them interact with the Electric Field  $\vec{E}$ , and the Magnetic Field  $\vec{B}$ .

Each of these fields is described by a vector at each point in space, describing how strong (and in which direction) electricity and magnetism point there.

#### PIC MAGNETIC FIELD

But how *does* a charged particle interact with these fields? The right answer turns out to be the Lorentz Force Law:

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

Where  $q$  is the particle's charge and  $v$  is the particle's velocity!

In general particles are exposed to both electric and magnetic fields, but to give us some practice working with parametric curves, we will consider the two cases separately.

#### 8.3.1. Charges in an Electric Field

First, consider the case that we have a particle in a constant electric field

$$\vec{E} = \langle E_x, E_y, E_z \rangle$$

How does the particle behave? Recalling that  $F = ma$  and  $a$  is the second derivative of position, this gives us a system of equations:

$$m \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = q \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

Dividing by  $m$ , we see that this system of equations just says that the acceleration of  $x$ ,  $y$ , and  $z$  are all constant! But we know how to solve this type of equation: we saw it with our earliest gravity model. For instance, in the  $x$ -direction we have

$x'' = -\frac{qE_x}{2m}t^2 + v_x t + x_0$  And similarly for the  $y$  and  $z$  components. Thus we can predict the path of particles in an electric field: they get accelerated (speed up), and follow parabolas!

### 8.3.2. Charges in a Magnetic Field

What about a particle in a constant magnetic field? Here the equation becomes  $m\vec{a} = q\vec{v} \times B$ . Setting  $B = \langle B_1, B_2, B_3 \rangle$ , we can write this equation out in full

$$m \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = q \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \times \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$$

If we multiply out that cross product, things will get messy quick and it will definitely look difficult to solve this equation! But - as we are imagining a constant magnetic field anyway - why don't we arrange it so that its pointing just along the  $z$ -axis? Then our magnetic vector becomes  $\langle 0, 0, B_z \rangle$ , and the cross product simplifies:

$$\vec{v} \times B = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x' & y' & z' \\ 0 & 0 & B_z \end{vmatrix} = \langle y'B_z, -x'B_z, 0 \rangle$$

Thus, in this special case we get the equations

$$m \begin{pmatrix} x'' \\ y'' \\ z'' \end{pmatrix} = qB_z \begin{pmatrix} y' \\ -x' \\ 0 \end{pmatrix}$$

The last of these equations is easy to solve:  $z'' = 0$  means that  $z(t) = v_z t + z_0$  as for our simple model of gravity. So,  $z$  travels linearly at a constant speed. But what do  $x$  and  $y$  do? This is a bit tricky, as all we know is how the second derivatives relate to the first!

To reduce some of the primes, let's write  $v_x$  and  $v_y$  for  $x'$  and  $y'$ . Then, dividing by  $m$  and setting the big constant  $qB_z/m = k$  to help simplify notation further, we have the system of equations

$$\begin{aligned} v_x' &= kv_y \\ v_y' &= -kv_x \end{aligned}$$

How can we solve these? Look what happens if we differentiate the second equation to get  $v_y'' = -kv_x'$ : now the right hand side has  $v_x'$  in it, so we can substitute the first equation in!

## 8. Differential Equations

$$v_y'' = -kv_x' = -k(kv)y = -k^2v_y$$

So, whatever  $v_y$  is, it's a function whose second derivative is a negative multiple of itself! We know functions like this - sine and cosine! More specifically,  $\sin(kt)$  and  $\cos(kt)$  since we need that  $k^2$  out front.

With this insight, we can quickly find a solution to our equations

$$v_x = \sin(kt) \quad v_y = \cos(kt)$$

But these aren't the solutions we are after - remember these are  $x'$  and  $y'$  - we want  $x$  and  $y$ ! So, we must integrate once more:

$$x = \int x' dt = \int \sin(kt) dt = \frac{-1}{k} \cos(kt) + x_0$$

$$y = \int y' dt = \int \cos(kt) dt = \frac{1}{k} \sin(kt) + y_0$$

Putting it all together, what do we have?

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{-\cos(kt)}{k} \\ \frac{\sin(kt)}{k} \\ v_z t \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix}$$

What is this? This is a helix!! So **charged particles move on helices through magnetic fields**. What are the radius of the circles that this helix traces out? They're  $1/k$  where remember  $k = qB_z/m$  is our constant. But this means that  $k$  is the *curvature* of these circles! (Recall last lecture - we derived that the curvature of a circle was the reciprocal of its radius)

This gives us another piece of information to understand particles: **the curvature of the circle a charged particle traces out in a magnetic field is  $qB/m$**  If we know the magnetic field and either the mass or the charge - we can solve for the other so long as we can measure the radius of the helix! This is a way to actually measure the mass of elementary particles!

But there's still one secret this equation has yet to give up: our constant  $k$  here is a multiple of the charge, and it also shows up *inside* the sine and cosine. What happens to a parametric curve like  $(\cos t, \sin t)$  when you multiply  $t$  by a negative number? It reverses in direction! Thus, the direction a particle traces out a helix depends on its charge: **positively and negatively charged particles spiral in different directions in the same magnetic field**

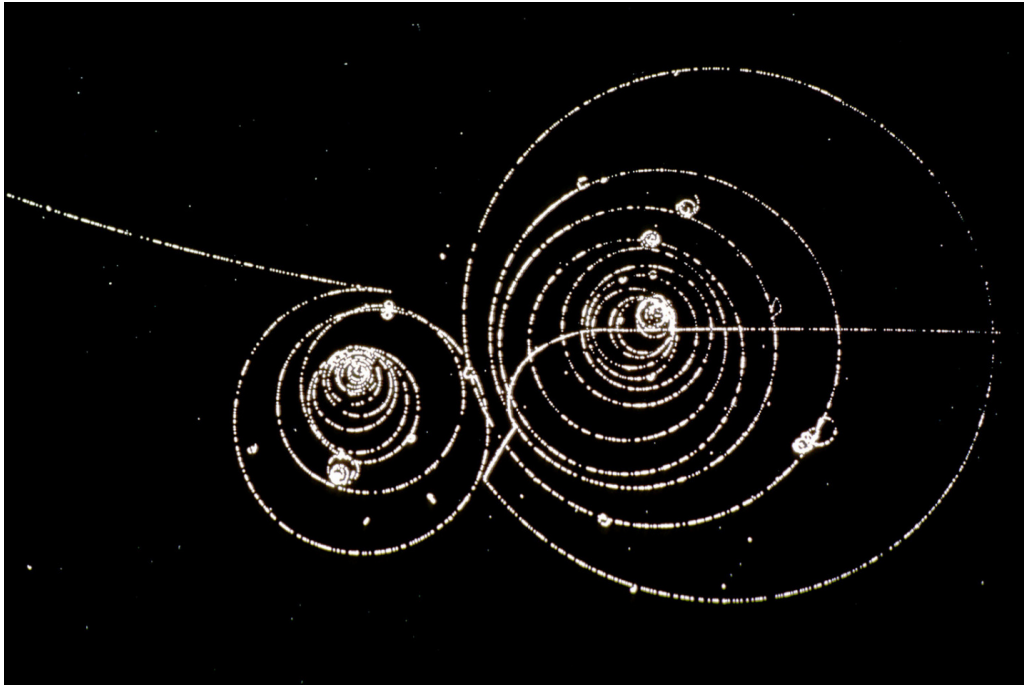


Figure 8.1.: Two particles tracing out helices in a magnetic field. These spiral inwards as the particles are losing energy by bumping into air and vapor to leave these tracks in a bubble chamber.

## 8. Differential Equations

### 8.3.3. The Discovery of Antimatter

This is how we discovered antimatter!

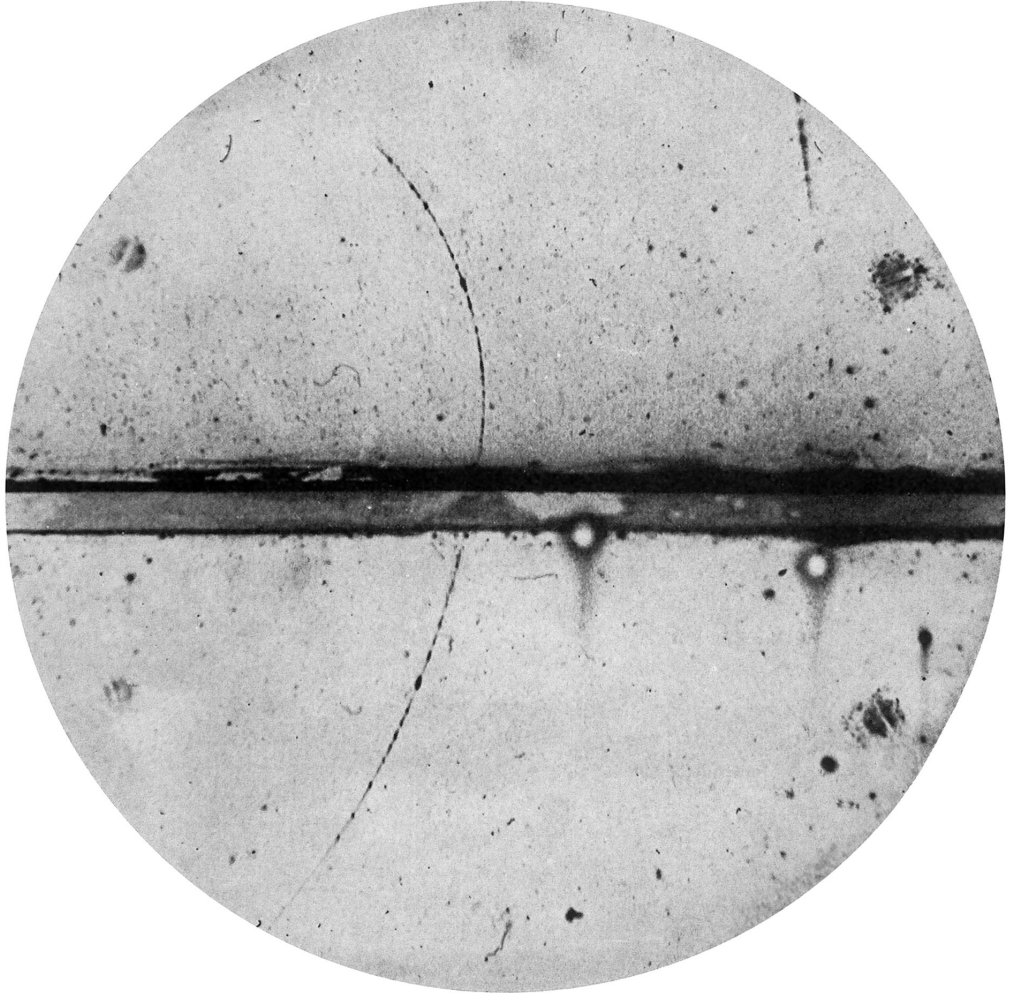


Figure 8.2.: The first piece of antimatter ever observed, which was noticed by this portion of a helical track having the same curvature as the electron's (so the same mass) but bending in the opposite direction (so, opposite charge)

## 8.4. Weather

(You are not responsible for this topic either)

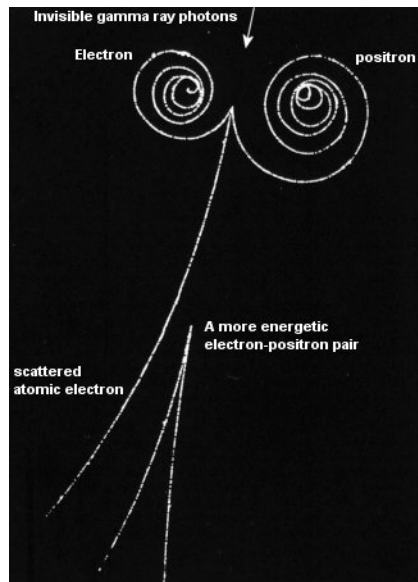


Figure 8.3.: An electron and a positron spiraling in opposite directions.

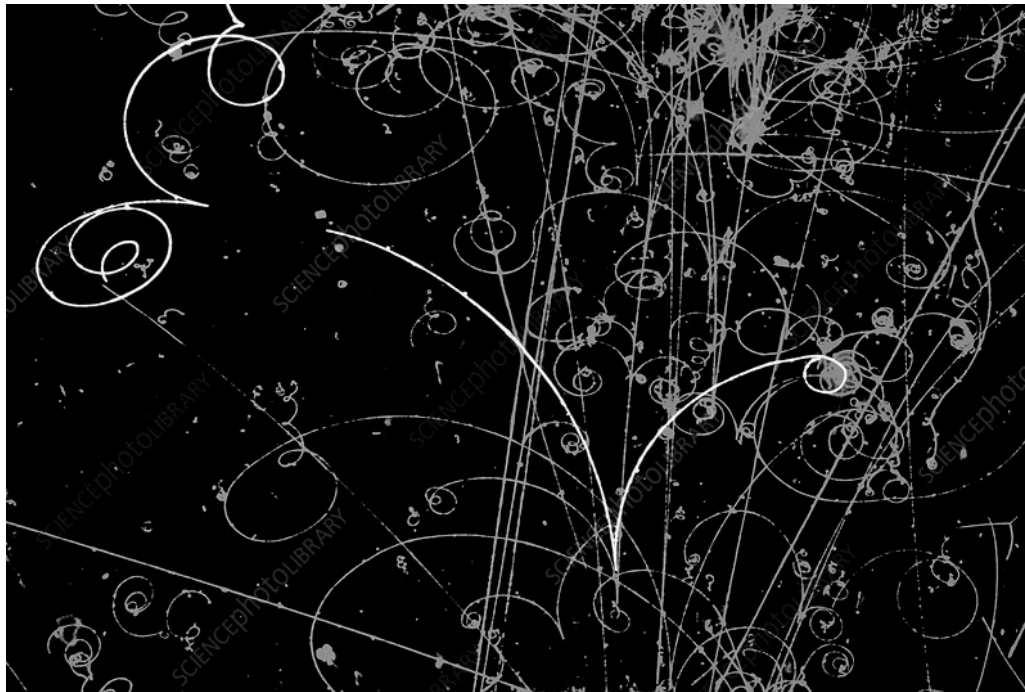
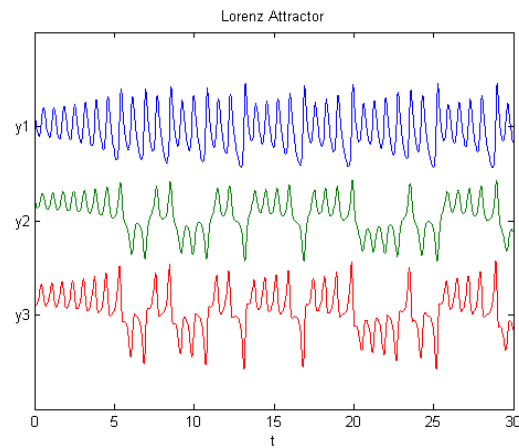


Figure 8.4.: How many particle/antiparticle pairs can you find in this image?

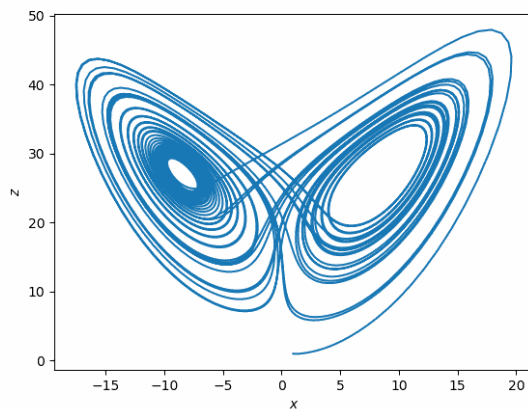
## 8. Differential Equations

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \sigma y - x \\ rx - y - xz \\ xy - bz \end{pmatrix}$$

Here are some solutions to the Lorenz equations of  $x, y, z$ .



Of course this is *much* easier to interpret what is going on if we instead plot these solutions as a vector curve in three dimensions!



Here's a program that solves the Lorenz equations in real time, so you can see an initially rather collected set of initial conditions quickly spread out all over the place and get mixed up: this is why it is hard to predict the weather!

<https://stevejtrettel.site/code/2023/lorenz-attractor>



**Part III.**

# **Multivariate Differentiation**



## 9. Introduction

So far in this course we have been studying parametric curves. These are functions which have a *single input* (the parameter) and multiple outputs (perhaps  $x, y, z$  or a color for each pixel of an image, etc.) Now we turn to reverse the situation, a function which has *multiple inputs* but a *single output*! These are sometimes called multivariable functions (to emphasize the number of inputs), but also are called scalar functions (to emphasize the output is a single number, or a *scalar*).

In physics, functions with multiple inputs are often called fields. So yet another name for this class of things is a *scalar field*! Just as I have done in these notes elsewhere (with vector notation, for example) I will try to use all of these terms to help you prepare for the real world where there's no standardization.

**Definition 9.1** (Multivariate Function). A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  taking in an  $n$ -tuple of real numbers, and outputting a single real number.

such a function is written  $f(x, y, z, \dots)$  for example

$$f(x, y, z) = 2x^2 - 3x + \sin(e^z)$$

The **domain** of a scalar function is the set of inputs for which the function makes sense.

**Example 9.1.** The domain of  $f(x, y) = \ln(x^2 + y^2 - 1)$  is the *exterior* of the unit circle in the plane  $D = \{(x, y) \mid x^2 + y^2 > 1\}$ . This is because the logarithm of zero or a negative number is underfined, so the only allowable inputs are when  $x^2 + y^2 - 1$  is positive.

Our goal in this chapter is to get comfortable with multivariable functions from two different perspectives: drawing graphs and drawing level sets.

## 9. Introduction

### 9.0.1. Examples

Scalar fields show up everywhere in mathematics and the sciences. Consider the temperature in a room: this is a function that takes in a point (say, with three coordinates  $(x, y, z)$ ) and returns a single number - the temperature of the air at that point in space. This is then a function  $T : \mathbb{R}^3 \rightarrow \mathbb{R}$ .

If we wanted to track the temperature over time, this could be done with a function  $\mathbb{R}^4 \rightarrow \mathbb{R}$  which takes  $(x, y, z, t)$  and returns the temperature there.

Of course - temperature is just an arbitrary (but conceptually useful!) example. **Any** quantity you can measure at different points in space(time) gives a scalar function of 3 or 4 variables!

**Exercise 9.1.** Come up with some examples of scalar functions that you think about in daily life (without necessarily having ever thought about them mathematically!)

## 9.1. Graphs

The *graph* of a 1-variable function  $f$  is a curve in the plane: it's the set of points  $(x, y)$  where  $y = f(x)$ . We can make a similar definition for the *graph* of a scalar function

**Definition 9.2.** The graph of a function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is a subset of  $\mathbb{R}^{n+1}$ : it's the set of points  $(\vec{x}, y)$  where  $\vec{x} \in \mathbb{R}^n$  and  $y = f(\vec{x})$ .

This isn't that useful for *visualizing* unless dimensions are pretty small! If there are  $n$  inputs and 1 output we need  $n + 1$  dimensions to put them all on a graph. And, if we want to actually see the graph it better fit inside of 3D space so  $n$  can only be 1 (the case we already know!) or 2.

If  $n = 2$ , then we have two input variables and it's easiest to think of them as being on the  $xy$  plane. Then we may name the output variable  $z$ , and plot our function  $z = f(x, y)$  as the points  $(x, y, f(x, y))$  in 3D space.

Here's a graphing calculator for functions  $z = f(x, y)$ :

<https://stevejtrettel.site/code/2023/graph3d>

### 9.1.1. How to Draw Graphs

Drawing a 3D graph is difficult to do in general: but often we can use our knowledge of 2D graphs to try and build up an understanding by **slicing**. The idea is to take a function  $z = f(x, y)$  and plug in *constant* values for one of the variables, then try to stack these slices to get a model of the entire surface.

**Example 9.2** (Slicing  $z = xy$ ). Draw the slices of  $z = xy$  - When  $y = 0$  this is the horizontal line  $z = 0$  - When  $y = 1$  this is the line  $z = x$  - When  $y = 2$  this is the line  $z = 2x$

Thus, as  $y$  increases, the *slope* of the graph's cross section increases!

**Example 9.3** (Slicing  $z = ye^x$ ). Here the slices for  $y = 0$ ,  $y = 1$  and  $y = 2$  look like  $0 e^x$  and  $2e^x$ . They are copies of the exponential getting steeper and steeper.

**Example 9.4** (Slicing  $z = x^2 + y^2$ ). Slices of this function are parabolas in both the  $x$  and  $y$  directions. Fixing  $y$  equal to different values, the parabolas shift upwards! For  $y = 0, 1, 2$  we get

$$z = x^2, \quad z = x^2 + 1 \quad z = x^2 + 4$$

### 9.1.2. Useful Graphs to Know

There are a couple multivariable functions whose graphs are good to know. Indeed - we've already met some of these before! If  $f(x, y)$  is a *linear* equation like  $ax + by + c$ , the graph is the set of points

$$z = ax + by + c$$

This is a plane! (It might help to rewrite as  $ax + by - z = -c$  to see its an implicit equation, with normal vector  $\langle a, b, -1 \rangle$ .)

What about  $z = x^2 + y^2$ ? We saw this one above in our discussion of slicing, and we also saw it earlier (in "Shapes"): its a parabola of revolution!

Another useful function to know is the *saddle surface*  $z = x^2 - y^2$ . We also met this one back in the discussion of "Shapes".

9. Introduction

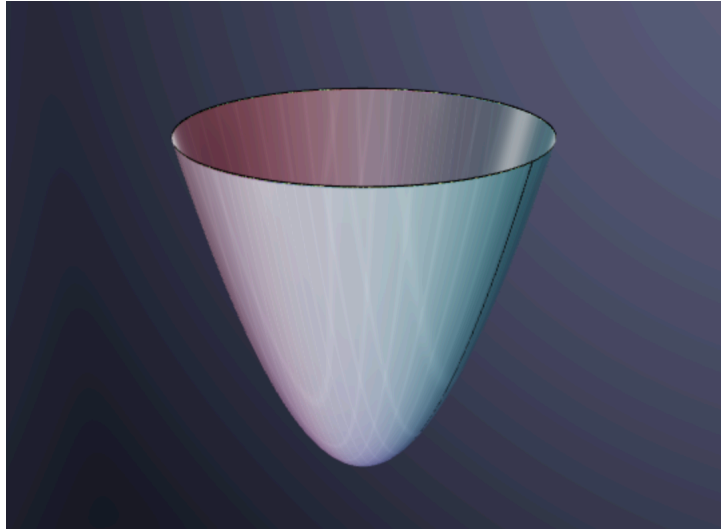


Figure 9.1.: The graph of  $z = x^2 + y^2$

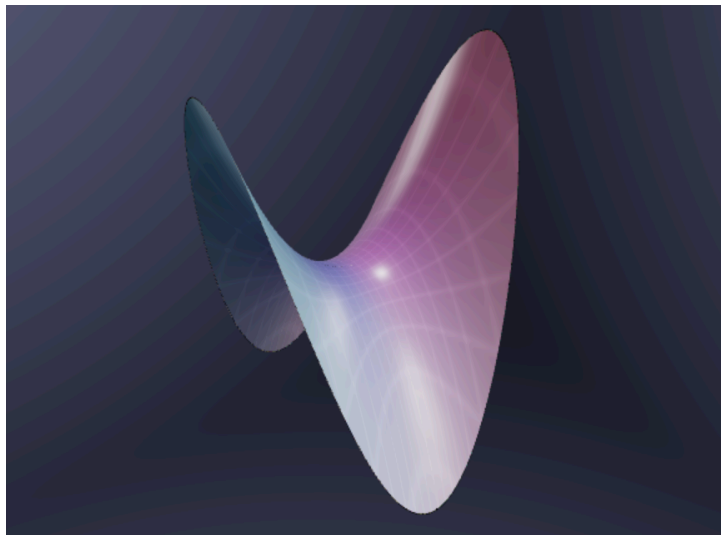


Figure 9.2.: The graph of  $z = x^2 - y^2$

## 9.2. Level Sets

Above we looked at one means of drawing a graph by slicing: we attempted to slice it by *vertical planes* into the graphs of simpler 1-dimensional functions that we already knew! This works sometimes, but if you can't quickly stack the slices into a coherent image in your mind, knowing the slices won't help you with much else.

So here, we seek other methods of understanding these functions, beyond their graphs. By far the most useful way to depict multivariable functions is by instead slicing with *horizontal planes* and drawing their level sets.

**Definition 9.3** (Level set). The level set corresponding to  $c \in \mathbb{R}$  for a function  $f(x, y)$  is the set of points in the plane that  $f$  maps to  $c$ :

$$L_c = \{(x, y) \mid f(x, y) = c\}$$

Here's a graphing calculator that will draw for you the level sets of a function: I often think about "sea level" when I think of a level set - a coastline is the level set  $h(x, y) = 0$  above the water. And different level sets correspond to what the coastlines *would be* if the sea was different heights.

<https://stevejtrettel.site/code/2023/contour-slicing>

A *contour plot* is a drawing of the *domain* of a function, with level sets representing various values of the *range*. These are perhaps most familiar from elevation maps. Drawing multiple level sets at once gives a good sense of the behavior of the entire function, though its most effective when the individual level sets are *labeled* somehow (often by color) so you can get a sense of their relative values.

Below, we draw the same map but only plot the level curve corresponding to 200ft above sea level.

$$L_{200} = \{(x, y) \mid h(x, y) = 200\}$$

A single level set of a function is actually something that you've encountered in previous courses: we called it an *implicit equation* as it defines a shape implicitly by saying  $(x, y)$  lives on the curve if  $h(x, y)$  has a specific value - instead of specifying it *explicitly* like a function.

Below is a graphing calculator to let you draw some contour plots. Can you imagine the 3D shape from its 2D slices?

<https://stevejtrettel.site/code/2023/levelsets>

9. Introduction

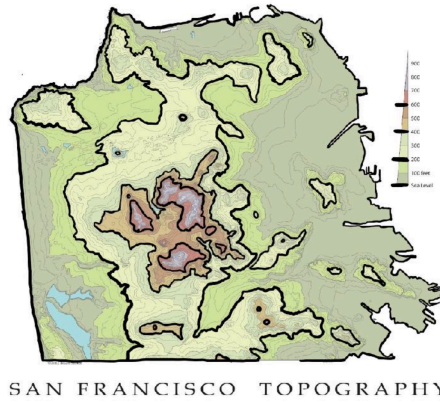


Figure 9.3.: A topographic map of San Francisco



Figure 9.4.: Some Level curves of the map. Can you see lone mountain?



Figure 9.5.: The 200ft-level set



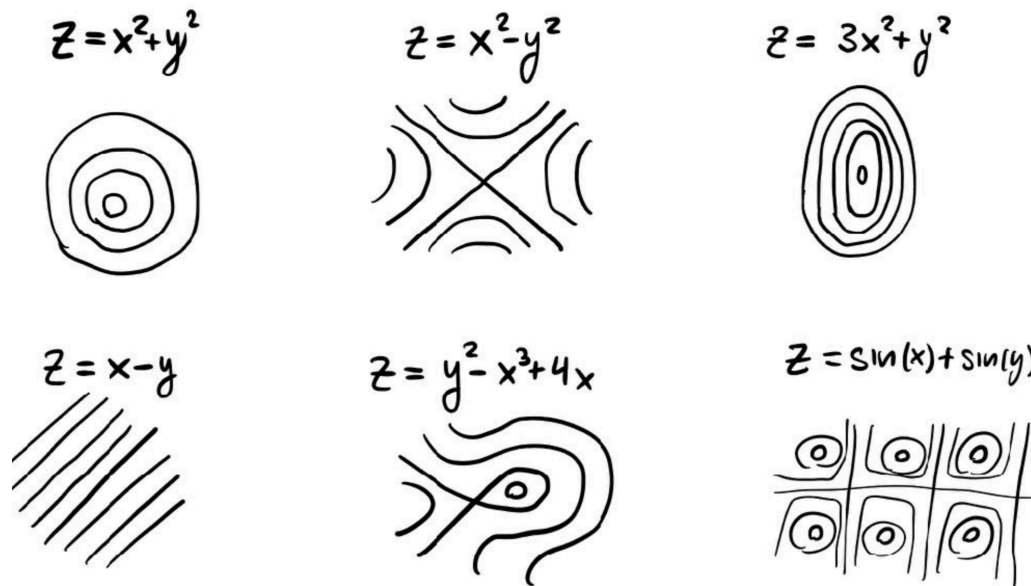


Figure 9.6.: Some contour plots of various functions.

After looking at several functions and their level sets, you'll start to notice that there are a couple "important behaviors" that show up again and again. These are

- Concentric rings, around a point
- Nearly parallel lines
- Two lines crossing each other.

These signify three important types of behavior, which we can see by looking back to our "useful graphs to know"

Concentric rings around a point signifies the function has either a maximum or a minimum there.

Nearly parallel lines means the function is increasing or decreasing there.

Two lines crossing means that we are at a *saddle shaped point* on our graph - it increases in two directions and decreases in the other two.

It turns out, that all behavior of level sets is built out of these three basic behaviors.

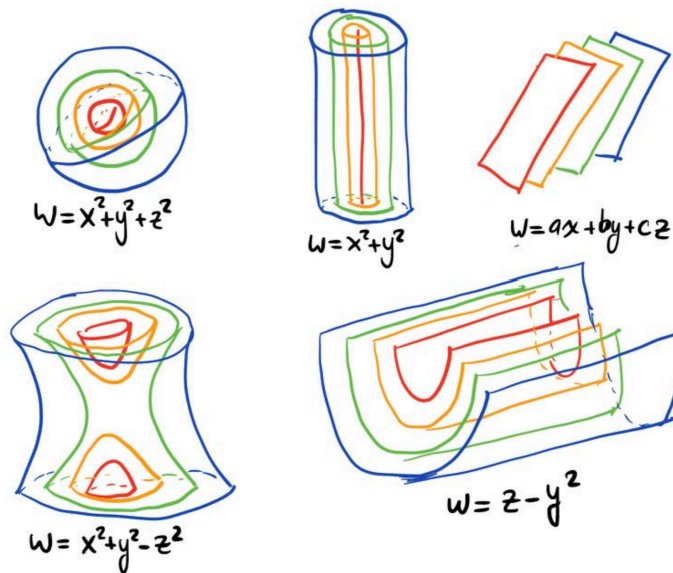
### 9.3. Functions of $\geq 3$ Variables

Drawing a contour plot is a form of *dimension reduction*: we've managed to understand the behavior of a function  $f$  whose graph  $z = f(x, y)$  lies in 3 dimensional space, by only looking at a 2-dimensional image (its domain, covered in level sets).

This technique can help us level up our intuition to functions of *three variables*: things like  $w = f(x, y, z)$  whose graphs would naturally live in four dimensional space!

**Exercise 9.2.** What do the level sets of the function  $w = x^2 + y^2 + z^2$  look like, for different values of  $w$ ?

In three dimensions there are more types of *basic behavior* than the ones we saw in 2D. You don't need to learn all of them: but to try and get some intuition for the fourth dimension its a good exercise to try and imagine what the graphs of these functions must be like, if their contours are drawn below.



Three dimensional level sets describe *implicit surfaces* which are extremely useful objects. As we've already seen with curves, sometimes writing down a parameterization can be hard. And this is even more true for surfaces! So having another way to express complicated ones can be a huge help. Below are two examples where I have used implicit surfaces in mathematical rendering.

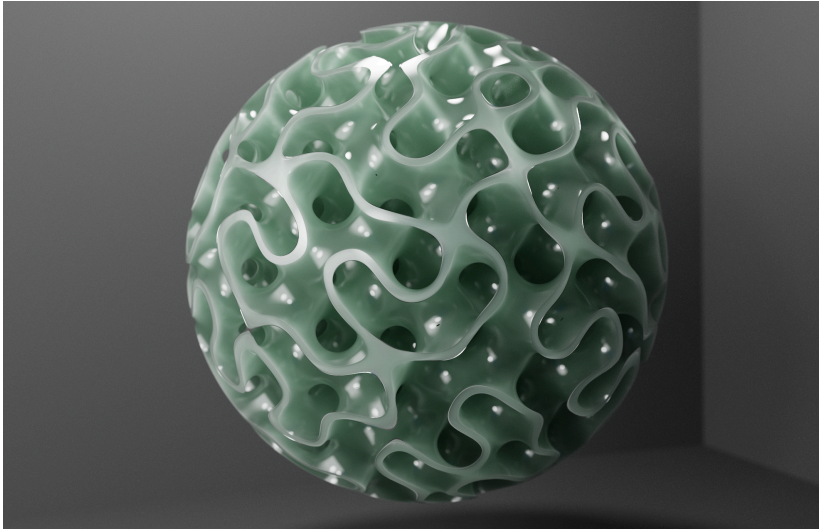


Figure 9.7.: The implicit surface  $\sin(x)\cos(t) + \sin(y)\cos(z) + \sin(z)\cos(x) = 0$ . This is a good approximation to a particularly important mathematical surface called the *gyroid*.

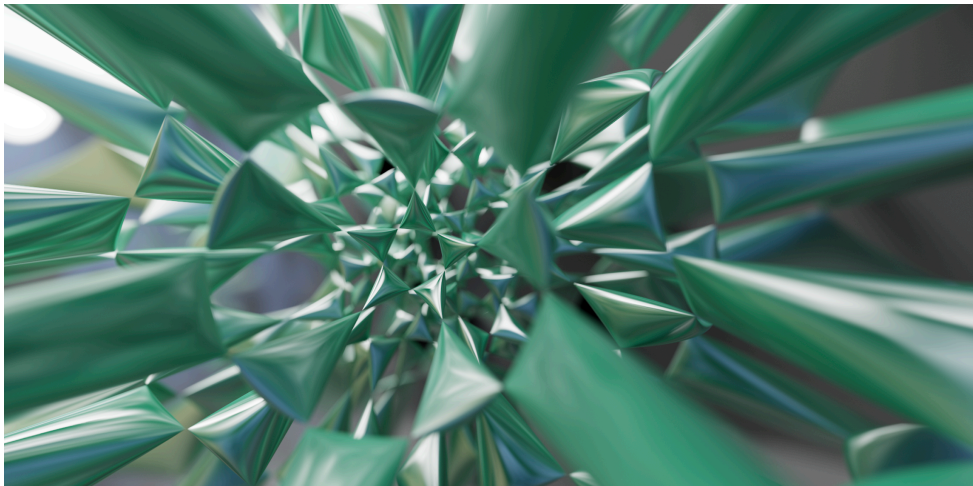


Figure 9.8.: This surface is called the *Barth Decic* and arises in a field of mathematics called algebraic geometry. Its equation is given below

## 9. Introduction

$$\begin{aligned} & 8(x^2 - p^4y^2)(y^2 - p^4z^2)(z^2 - p^4x^2)(x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2) \\ & + a(3 + 5p)((x^2 + y^2 + z^2 - a))^2((x^2 + y^2 + z^2 - (2 - p)a))^2 \\ & = 0 \end{aligned}$$

### 9.4. Videos

The Calculus Blue Series introduction to multivariate functions:

<https://youtu.be/e2TJpv39SJo?si=mGoSRbZ7BOqstCZi>

<https://youtu.be/gbj0YfdovxU?si=7t8M9qKPA8JkPp4U>

The Calculus Blue series on Level Sets:

<https://youtu.be/zEvGUXW1BI?si=54UKw1EULxEsU6IV>

## 10. Partial Derivatives

How do we differentiate a function with multiple inputs? We will learn several ways to do this throughout the course. But all methods rely on one fundamental idea: the *partial derivative*.

### 10.1. Geometry of Partial Derivatives

If we take a function  $f(x, y)$  and hold the  $y$  value constant, we get a function of just  $x$ . For example, if  $f(x, y) = \sin(x * y)$  and  $y = 2$  we get the function  $f(x, 2) = \sin(2x)$ . This sort of function we already know how to take the derivative of!

$$f(x, 2)' = \frac{d}{dx}f(x, 2) = 2 \cos(2x)$$

What does this derivative *mean*? Well, we are measuring the slope *in the x-direction* along the line where  $y = 2$ . Here's a graphing calculator showing this, for different slices and different points!

<https://stevejtrettel.site/code/2023/partial-derivatives>

What happens if we instead looked at the slice where  $y = 7$  Then we'd have  $f(x, 7) = \sin(7x)$  and the derivative would be

$$f(x, 7)' = \frac{d}{dx}f(x, 7) = 7 \cos(7x)$$

Thus, whatever  $y$  is we see it comes out front as a coefficient via the chain rule. This tells us that if we just call the constant  $y$  (and don't bother to specify its numerical value) we should get

$$f(x, y)' = \frac{d}{dx}f(x, y) = y \cos(xy)$$

## 10. Partial Derivatives

The only thing confusing here is that unless we know what we are doing, it's hard to tell what the prime means. So we should probably not use this notation when there's more than one variable.

In fact, to signify that we are taking the derivative of a *multivariable function*, its customary to write the  $d$  a little fancy as well, using the italic  $\partial$

**Definition 10.1** ( $x$ -Partial Derivative). If  $f(x, y, z, \dots)$  is a function of multiple variables, the  $x$ -partial derivative with respect to  $x$  is the result of treating all other variables as constants, and differentiating with respect to  $x$ . It's denoted

$$\frac{\partial f}{\partial x} = \lim_{h \rightarrow 0} \frac{f(x+h, y, z, \dots) - f(x, y, z, \dots)}{h}$$

Partial derivatives are ubiquitous in the sciences, and because they are so widely used in so many fields, there are also many common notations for them. I will use many of the notations interchangeably in class, to prepare you for the real world: and for reference the most common ones appear below.

**Definition 10.2** (Notations for Partial Differentiation). The partial derivative of  $f$  with respect to  $x$  may be written as

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f = \partial_x f = f_x$$

The last notation takes some getting used to at first: *the subscript represents differentiation!* But because of its conciseness, it is very commonly used when performing calculations.

### 10.1.1. Implicit Partial Derivatives

Just like in single variable calculus, we can use *implicit differentiation* to take the derivative of implicit equations with multiple variables.

### 10.1.2. Implicit Differentiation

**Example 10.1** (Implicit Partial Differentiation). Find the derivative  $\partial_x z$  of the expression

$$x^3 + y^3 + z^3 + 6xyz = 7$$

Here we act as though  $z$  is implicitly a function of  $x$ , and we differentiate the whole equation:

$$\frac{\partial}{\partial x}(x^3 + y^3 + z^3 + 6xyz) = \frac{\partial}{\partial x}7$$

Computing this (where we need the product rule on the last term, since we have both an  $x$  and a  $z$ -where  $z$  is implicitly a function of  $x$ !) gives

$$3x^2 + 0 + 3z^2 \frac{\partial z}{\partial x} + 6yz + 6xy \frac{\partial z}{\partial x} = 0$$

Then, we just solve for  $\partial_x z$ :

$$(3z^2 + 6xy) \frac{\partial z}{\partial x} = -3x^2 - 6yz$$

$$\frac{\partial z}{\partial x} = \frac{-3x^2 - 6yz}{3z^2 + 6xy}$$

What is implicit differentiation *measuring*? Think back to the 2-dimensional case, from calculus 1. There we had an implicit equation - what we now know to be a *level set* - and we were trying to measure  $dy/dx$ : that is, for a small change in  $x$ , how much does  $y$  have to change, to stay on the level set?

The same picture works here, but now in higher dimension. An implicit equation in  $x, y, z$  determines a surface in  $\mathbb{R}^3$  - which is the level set of some 3-variable function. And a quantity like  $\partial z/\partial x$  tells us how much  $z$  has to change to stay on the surface, if we change  $x$  a little bit.

We will develop more sophisticated versions of this when we talk about linearization and the gradient.

## 10.2. Higher Derivatives

Higher order partial derivatives are no more difficult: each time you take the derivative, you just treat all other variables as constants.

For instance, the second partial  $x$  derivative is just what you get by taking the  $x$  derivative twice:

$$\partial_x \partial_x (\cos(xy)) = \partial_x (-y \sin(xy)) = -y^2 \cos(xy)$$

But you can also take partials with respect to different variables.

$$\partial_x \partial_y (x^3 y^2) = \partial_x (2x^3 y) = 6x^2 y$$

**Definition 10.3.** A *higher partial derivative* is just the result of taking the partial derivative more than once (perhaps with respect to different variables). When doing this, one needs to be careful with notation: the “derivative notations” are all read like function composition

$$\partial_x \partial_y \partial_z f = \frac{\partial}{\partial x} \frac{\partial}{\partial y} \frac{\partial}{\partial z} f$$

both mean *do the  $z$  partial, then the  $y$  partial, then the  $x$  partial.*

The *subscript notation* is read from inside out:

$$f_{zyx} = ((f_z)_y)_x$$

is the equivalent to the above: doing  $z$  first, then differentiating with respect to  $y$ , and finally with respect to  $x$ .

**Theorem 10.1** (Equality of Mixed Partials). *So long as the partial derivatives are defined and continuous, the order in which you take them does not matter.*

$\partial_x \partial_y f = \partial_y \partial_x f$  This works with higher order derivatives as well

$$f_{xyxx} = f_{xxyx} = f_{yxxx} = \dots$$

In fact, the second order partial derivatives together form an operation called the *laplacian* which arises in many applications:



### 10.3. Partial Differential Equations

**Definition 10.4.** The Laplacian operator is the sum of the (non-mixed) second order partial derivatives: it is sometimes written as  $\Delta$  and sometimes as  $\nabla^2$ : in the plane this is

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and in higher dimensions, its analogous just with more variables. To take the laplacian of a function, you just find its non-mixed second partials, and add them all up:

$$\Delta f = f_{xx} + f_{yy}$$

One way to imagine what the laplacian is measuring is a sort of *average concavity*: it adds up the concavity in both the  $x$  and  $y$  directions. Thus, a function like  $x^2 + y^2$  has laplacian

$$\Delta(x^2 + y^2) = 2 + 2 = 4$$

so its concave up on average,  $-(x^2 + y^2)$  has laplacian  $-4$  so its concave down, and  $x^2 - y^2$  has laplacian equal to zero: it is concave up in one direction and concave down in the other: so added together they cancel.

Functions whose Laplacian are zero are called *harmonic functions* and play a huge role in understanding differential equations, physics, and engineering.

## 10.3. Partial Differential Equations

Partial derivatives are the language in which much of modern science is written. We saw in the last portion of the course that *vector valued differential equations* are the right language to describe the motion of *single particles*: but what about quantities that depend on more than one variable?

A first example is simply waves on a string: when a guitar string is pulled taught, if you try to pluck it away from rest it *pulls back on you* - the farther you pull it away, the harder it pulls back.

The amount a string curves away from its straight line equilibrium is captured (roughly) by its concavity. And so one simple model of string motion would say the bigger the concavity the faster it wants to “snap back”. Said more precisely

## 10. Partial Derivatives

### The acceleration of the string is proportional to its concavity

Writing this in math - if the string's displacement at position  $x$  and time  $t$  is given by the function  $W(x, t)$ , we have the partial differential equation below:

$$\partial_{xx}W = \partial_{tt}W$$

In two dimensions, a wave equation measures the displacement of a circular membrane, like a drumhead or the interior speaker of an earbud. Here we have to account for displacements in both the  $x$  and  $y$  directions. Thus, the 2 dimensional wave equation is below (now written in the more 'verbose' notation for partial derivatives)

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} = \frac{\partial^2 W}{\partial t^2}$$

Here's some solutions to this equation:

<https://stevejtrettel.site/code/2022/circular-drum>

This same wave equation in three dimensions describes the propagation of *electromagnetic waves* - or light! This was a triumph of 19th century physics, where James Clerk Maxwell derived this wave equation from his equations for the electromagnetic field. Below, you can see a (numerically computed) solution to this equation, showing a light beam being focused by a glass lens.

<https://stevejtrettel.site/code/2021/wave-eqn-flash>

Similar partial differential equations occur throughout physics and engineering. (If you have yet to be convinced of the wide applicability of partial derivatives, look up "continuum mechanics" and try to find a topic you're interested in.)

One final example I'll mention here is quantum mechanics, where the fundamental equation (called the Schrodinger equation) is a replacement of Newtons Law ( a vector valued differential equation) with a partial differential equation. This big change in the mathematics is what causes people to say that quantum particles can be "like waves".

Below is a calculator I wrote for playing around with the "double slit experiment" in quantum mechanics. This will not come up further in our course, but feel free to ask me if you are interested!

<https://stevejtrettel.site/code/2021/schrodinger-double-slit>

## 10.4. Videos

### 10.4.1. Calculus Blue

[https://youtu.be/78bXzF\\_J3RM?si=AdXYWkEEwH0tkSOX](https://youtu.be/78bXzF_J3RM?si=AdXYWkEEwH0tkSOX)

[https://youtu.be/V-\\_WuybYkyg?si=9hC-AnNP52ehPfcZ](https://youtu.be/V-_WuybYkyg?si=9hC-AnNP52ehPfcZ)

[https://youtu.be/T\\_7FXwImddU?si=ex91sAxbEjLqWZJC](https://youtu.be/T_7FXwImddU?si=ex91sAxbEjLqWZJC)

[https://youtu.be/DYrs\\_cOg0E8?si=V3bmWBMgCQrFhuHs](https://youtu.be/DYrs_cOg0E8?si=V3bmWBMgCQrFhuHs)

### 10.4.2. Khan Academy

[https://youtu.be/AXqhWeUEtQU?si=XRKrVYhjZC\\_qWp-p](https://youtu.be/AXqhWeUEtQU?si=XRKrVYhjZC_qWp-p)

<https://youtu.be/dfvnCHqzK54?si=rasi8aG3Gxt0LFIM>

<https://youtu.be/J08-L2buigM?si=tNUmEbz-7pFXO2N->

### 10.4.3. Example Problems:

[https://youtu.be/btcSjC5z7WQ?si=SnKosx3\\_t96cjni3](https://youtu.be/btcSjC5z7WQ?si=SnKosx3_t96cjni3)

<https://youtu.be/3itjTS2Y9oE?si=7rWWzXWUHRpllUUC>

[https://youtu.be/EoEV5-\\_mLeM?si=LkKZqesDmHVozqcy](https://youtu.be/EoEV5-_mLeM?si=LkKZqesDmHVozqcy)



# 11. Linearization

## 11.1. The Fundamental Strategy of Calculus

Take a complicated function, zoom in, replace it with something linear. Thus we are looking to replace graphs by planes. We have two formulas for planes (implicit and parametric) so we can have two ways to do this.

First we will look at the more “implicit” approach, and produce an equation of a plane of the form  $z = ax + by + c$  (this is just re-arranged from our most general form in the previous chapters)

To start, look back on the tangent line formula from single variable calculus:

$$y = y_0 + f'(x_0)(x - x_0)$$

Our extension to multiple dimensions is just...to add more variables! We need to adjust  $z$  not just for changes in  $x$  any more, but also for changes in any input.

**Theorem 11.1** (Tangent Plane).

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Same formula works in higher dimensions, by just adding more terms. Since this gives us an implicit plane, we can re-arrange this to the “standard form” and find the normal vector to the graph

**Theorem 11.2** (Normal Vector to a Graph). *At the point  $(x_0, y_0)$  the plane*

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - z = -z_0$$

*is tangent to the graph of  $f$ . Thus, the normal vector is the coefficient vector*

$$n = \langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \rangle$$

## 11. Linearization

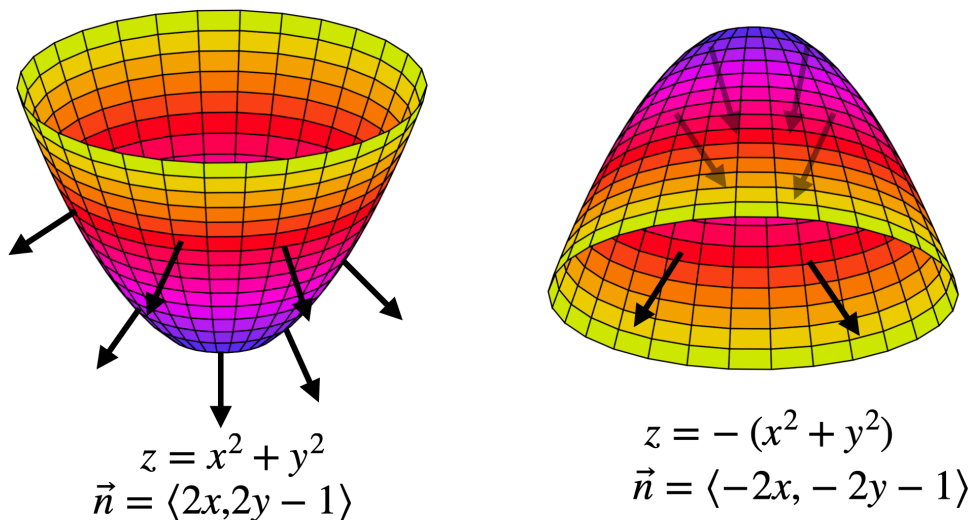


Figure 11.1.: The downward facing normal, on two graphs.

Note that any scalar multiple of this vector is *also* a normal vector to the graph - this just provides *one such vector*. And, since the  $z$  is downwards, this is the *downwards pointing normal vector*.

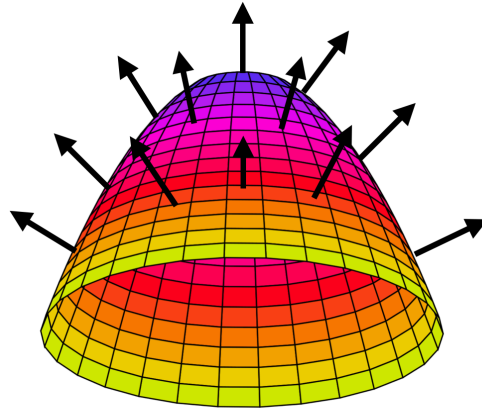
Depending on the application, sometimes we want the *upward facing normal*: that's what you'd get by multiplying this by  $-1$  so that the last coordinate is a 1:

We will use normal vectors to surfaces a lot in the last portion of this course, on Vector Analysis. Here we will often need to be careful, and thinking about whether we want the normal that is pointed up or down in a given application.

**Example 11.1.** Find the tangent plane to  $z = x^2 + 2y^2$  above the point  $(x, y) = (2, 3)$ .

The second method we could use is to try and find a *parametric equation* for the plane. Again, we start by considering the one dimensional case. A parametric line is of the form  $vt + p$  for  $v$  the direction and  $p$  a point. How do we find the tangent line to a function  $y = f(x)$ ? The derivative measures the *slope*, so if we go over by 1 unit in the  $x$  direction we go up by  $f'(x_0)$  units in the  $y$ -direction. Thus, the direction vector is

$$v = \langle 1, f'(x_0) \rangle$$



$$z = -(x^2 + y^2)$$

$$\vec{n} = \langle 2x, 2y, 1 \rangle$$

Figure 11.2.: The upward facing normal.

Now all we need is the position: but that's just  $(x_0, f(x_0))$  on the graph! Putting these together we get the tangent line

$$L(t) = \begin{pmatrix} x_0 \\ f(x_0) \end{pmatrix} + t \begin{pmatrix} 1 \\ f'(x_0) \end{pmatrix}$$

What do we do for a function of two or more variables? We just...add more variables!

**Theorem 11.3** (Parametric Tangent Plane).

$$\vec{L}(s, t) = \begin{pmatrix} x_0 \\ y_0 \\ f(x_0, y_0) \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ f_x(x_0, y_0) \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ f_y(x_0, y_0) \end{pmatrix}$$

**Example 11.2.** Find a parametric equation for the tangent plane to the saddle  $x^2 - y^2$  at the point  $(1, 1, 0)$ .

### 11.1.1. Differentiability

Partial derivatives give us a way to test if a function itself is *differentiable* in multiple variables.

## 11. Linearization

**Definition 11.1** (Multivariate Differentiability). A function  $f$  is differentiable at a point  $p$  if its tangent plane at  $p$  is a good approximation to its graph near  $p$ .

Functions we will see are mostly differentiable, but warning there are functions that are not. Luckily, there's an easy way to check

**Theorem 11.4** (Multivariate Differentiability). *A multivariate function is differentiable at a point, if and only if all of its partial derivatives exist and are continuous at that point.*

### 11.2. Differentials

The Fundamental Strategy of calculus is to take a complicated nonlinear object (like a function that you encounter in some real-world problem) and zoom in until it looks linear. Here, this zooming in process is realized by finding the tangent plane. Close to the point  $(x_0, y_0)$  the graph of the function  $z = f(x, y)$  looks like

$$L(x, y) = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Where  $z_0 = f(x_0, y_0)$ . This is just rehashing our definition of the tangent plane of course: but one use for it is to be able to approximate the value of  $f(x, y)$  if you know the value of  $f$  at a nearby point  $(x_0, y_0)$  and also its partial derivatives there.

**Example 11.3.** Find approximate value value of  $x^2 + 3xy - y^2$  at the point  $(2.05, 2.96)$ .

Using linearization to estimate changes in a value: fundamental to physics and engineering. In 1-dimension, we define a variable called  $dx$  that we think of as measuring small changes in the input variable, and  $dy = y - y_0$  which measures small changes in the output. These are related by

$$dy = f'(x)dx$$

So, any change in the input is multiplied by the derivative to give a change in the output. We can do a similar thing in more variables. For a function  $f(x, y)$ , we have the tangent plane above

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Subtracting  $z_0$  and setting  $dz = z - z_0$ , and similarly for  $dx, dy$  we can rewrite this as below:



**Definition 11.2** (Differentials).

$$dz = f_x(x, y)dx + f_y(x, y)dy$$

This allows us to easily *estimate* how much  $z$  could change if we know how much  $x$  and  $y$  can change. This is of fundamental importance in *error analysis*, the foundation of all experimental science's ability to compare with theoretical predictions.

**Example 11.4.** The volume of a cone is given by  $V = \pi r^2 h/3$ . We have a cone which we measure the height to be 10cm and the radius to be 25cm, but our measuring device can have an error up to 1mm or 0.1cm. What is the estimated maximal error in volume our measurement could have?

This measurement can also be interpreted *geometrically*: this is the approximate volume of a thin-shelled cone of thickness 1mm with radius 25cm and height 10cm.

**Example 11.5.** The dimensions of a rectangular box are measured to be 75cm, 60cm and 40cm. Each measurement is correct to within 0.05cm. What is the maximal error in volume measurement we might expect?

## 11.3. Videos

### 11.3.1. Calculus Blue:

<https://youtu.be/4tNMHtMyDns?si=ITtSJE1NzBJ9bfMe>

[https://youtu.be/0vma0sbpSqU?si=07L\\_BNXg7pXjR5A8](https://youtu.be/0vma0sbpSqU?si=07L_BNXg7pXjR5A8)

<https://youtu.be/cK9fHRtv2Rk?si=16JIFRyPtoP3pN0V>

### 11.3.2. Khan Academy

[https://youtu.be/o7\\_zS7Bx2VA?si=pj3GE8x\\_iiOTBHRH](https://youtu.be/o7_zS7Bx2VA?si=pj3GE8x_iiOTBHRH)

### 11.3.3. Example Problems:

[https://youtu.be/oJ\\_LA1AYU18?si=UmgBmFOqecnmfub7](https://youtu.be/oJ_LA1AYU18?si=UmgBmFOqecnmfub7)

<https://youtu.be/cRGoTL00Ksg?si=0RWUC0TaeBVtuEWd>



## 12. The Gradient

### 12.1. Directional Derivatives

We have seen that  $\partial_x f$  and  $\partial_y f$  measure the slope of a multivariate function in the  $x$  and  $y$  directions, respectively. But what is its rate of change in the direction of an arbitrary unit vector  $u$ ?

**Definition 12.1** (Directional Derivative). The derivative of  $f$  in the direction of a unit vector  $u$  is denoted  $D_u f$  and is defined by the limit

$$\lim_{h \rightarrow 0} \frac{f(p + \epsilon u) - f(p)}{\epsilon}$$

Computing this seems difficult. But we can use the Fundamental Strategy of Calculus to save the day! We know that the slope in the direction  $u$  must lie on the tangent plane, which we have already parameterized in terms of the  $x$  and  $y$  partials:

$$\text{Plane}(s, t) = \begin{pmatrix} x \\ y \\ f(x, y) \end{pmatrix} + s \begin{pmatrix} 1 \\ 0 \\ \partial_x f \end{pmatrix} + t \begin{pmatrix} 0 \\ 1 \\ \partial_y f \end{pmatrix}$$

We can plug  $u = \langle a, b \rangle$  in here and take a look at the  $z$  coordinate to read off the change in  $z$ .

$$z = f(x, y) + a\partial_x f + b\partial_y f$$

So the *change in  $z$*  is just the quantity  $a\partial_x f + b\partial_y f$ . Alternatively, we can use the *differential* we derived from this linear approximation

$$dz = f_x dx + f_y dy$$

To see if  $dx = a$  and  $dy = b$  that  $dz = af_x + bf_y$ . It's just a linear combination of the two basic slopes we already know!

## 12. The Gradient

**Theorem 12.1** (Directional Derivative). If  $u = \langle a, b \rangle$  is a unit vector, then

$$D_u f(x, y) = a f_x(x, y) + b f_y(x, y)$$

All of this carries over to three or higher dimensions: if  $u = \langle a, b, c \rangle$  and  $f(x, y, z)$  is a three variable function then

$$D_u f = a f_x + b f_y + c f_z$$

### 12.2. The Gradient

Because the collection of partial derivatives  $\langle f_x, f_y \rangle$  will show up so often it will be useful to give this a name: the gradient.

**Definition 12.2** (The Gradient). The gradient of a function  $f(x, y)$  is

$$\nabla f = \langle f_x, f_y \rangle$$

The gradient of  $f(x, y, z)$  is the 3-dimensional vector

$$\nabla f = \langle f_x, f_y, f_z \rangle$$

**Definition 12.3** (Nabla). The symbol  $\nabla$  is called *nabla* or *del*, and is a shorthand for the *vector of partial derivative operators*:

$$\nabla = \langle \partial_x, \partial_y, \partial_z \rangle$$

This notation is convenient here as

$$\begin{aligned} \nabla f &= \langle \partial_x, \partial_y \rangle f \\ &= \langle \partial_x f, \partial_y f \rangle \\ &= \langle f_x, f_y \rangle \end{aligned}$$

But it will also be convenient later in the course, for defining other types of derivatives. A first benefit here is it can take our formula for the directional derivative and make it much simpler to remember!

**Theorem 12.2** (Directional Derivatives and the Gradient).

$$D_u f(x, y) = \nabla f \cdot \hat{u}$$

## 12.3. Geometry of the Gradient

Since we know the interpretation of dot products in terms of angles, we can use the directional derivative formula above to help us understand the direction the gradient points in.

If a vector  $u$  makes angle  $\theta$  with the gradient, we see the directional derivative in direction  $u$  is given by

$$D_u f = \nabla f \cdot u = \|\nabla f\| \|u\| \cos \theta = \|\nabla f\| \cos \theta$$

This actually tells us alot!

### Theorem 12.3.

- *The gradient points in the direction of maximal directional derivative.*
- *Its magnitude is the directional derivative in that direction*
- *In the orthogonal direction to the gradient, the directional derivative is zero: the function is not changing!*

<https://stevejtrettel.site/code/2022/gradient>

The last of these facts is so useful on its own, that it gets it's own theorem box:

**Theorem 12.4.** *The gradient vector is orthogonal to the level sets of a function, and points in the direction of increase.*

This is very helpful for understanding a function from its gradient, as it lets us convert between level set understanding and gradient understandings!

### 12.3.0.1. The Gradient and Level Sets

When level sets are close to each other, that means the function is steeply increasing or decreasing, so the gradient is *long*. When level sets are far apart, that means the function is only slowly changing, so the gradient is short. Thus, there's an inverse relationship between the length of the gradient and the density of level sets.

12. The Gradient

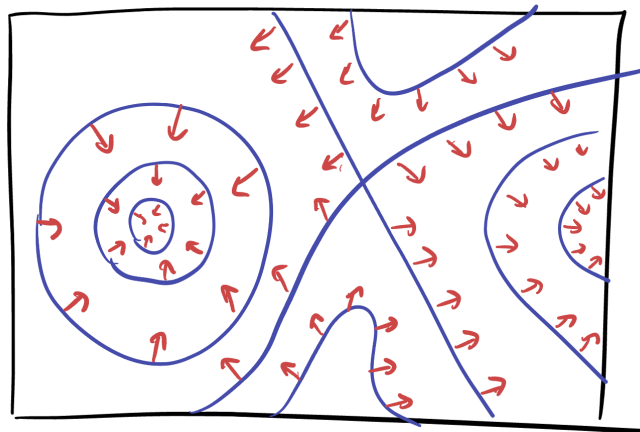


Figure 12.1.: The gradient is orthogonal to level sets.

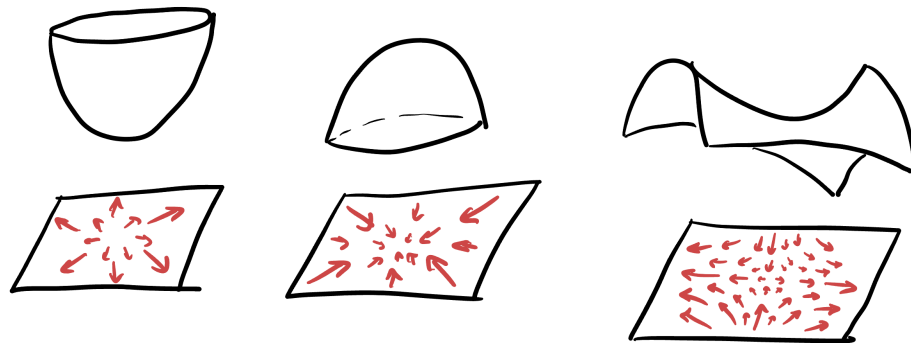


Figure 12.2.: The gradient points in the direction of steepest ascent.

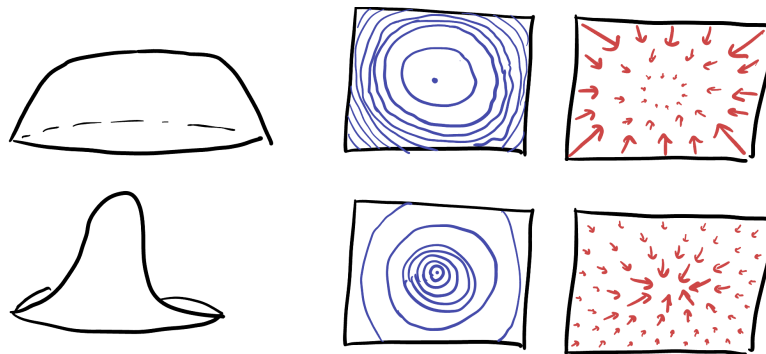


Figure 12.3.: The length of the gradient is inversely proportional to the density of contour lines.

### 12.3.1. Tangent Planes to Level Sets

Because the gradient is a *normal vector to level sets* we can use the gradient to derive the equation for a tangent plane to a surface! We previously wrote it down of for functions,

$$f_x(a, b, c)(x - a) + f_y(a, b, c)(y - b) + f_z(a, b, c)(z - c) = 0$$

But this was just in analogy with the tangent line case. Now, we wish to derive it from our original description of planes, in terms of their *normal vectors*: if  $p$  is a point on the plane and  $n$  is a normal vector to the plane, the equation

$$n \cdot ((x, y, z) - p) = 0$$

Describes the plane because it says  $(x, y, z)$  lies in the plane so long as the vector connecting it to  $p$  is orthogonal to  $n$ . Now that we know the gradient

$$\nabla f(a, b, c) = \langle f_x(a, b, c), f_y(a, b, c), f_z(a, b, c) \rangle$$

is the normal vector to our plane, we can directly write down the equation for the normal at  $(a, b, c)$ :

$$\nabla f(a, b, c) \cdot ((x, y, z) - (a, b, c)) = 0$$

and, after computing the dot product we can see it's the same equation we already know!

But knowing the normal vector also allows us to compute other geometric quantities of interest: such as the **normal line**: the parametric line which intersects a level set orthogonally.

This is also immediate: as if we know a point  $p$  and a direction vector  $v$ , the associated line is  $\ell(t) = p + tv$ . So here, the point is  $(a, b, c)$  and the normal vector is  $\nabla f(a, b, c)$  so the normal line is

$$\ell(t) = (a, b, c) + t\nabla f(a, b, c)$$

## 12. The Gradient

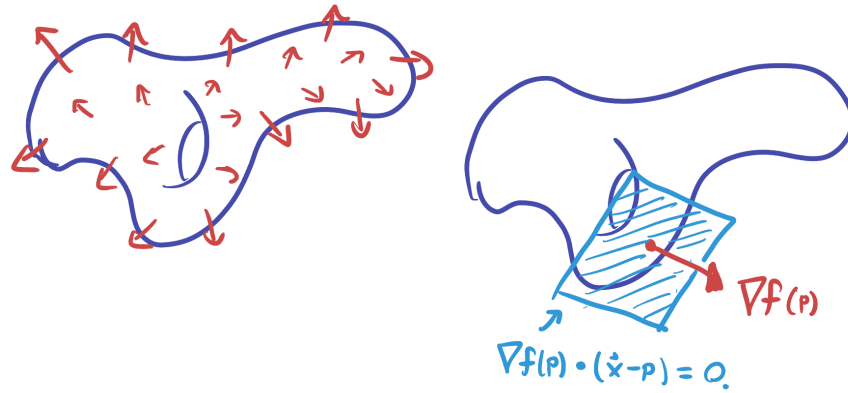


Figure 12.4.: The gradient is normal to level sets, even in 3D. This makes it easy to use the gradient to find the tangent plane to a level set.

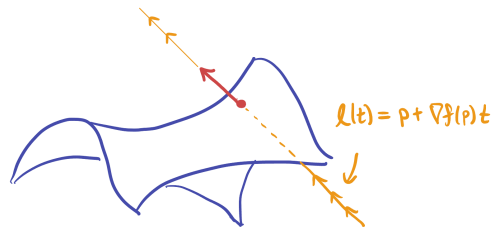


Figure 12.5.: The normal line to a level set in three variables.



**Example 12.1.** Compute the tangent plane and the normal line to  $x = y^2 + z^2 + 1$  at  $(3, 1, -1)$ .

First, we re-arrange so that the surface equation is written as a level set:  $x - y^2 - z^2 = 1$  with all the variables on one side. Now we can compute the gradient:

$$\nabla f = \langle 1, -2y, -2z \rangle \quad \nabla f(3, 1, -1) = \langle 1, -2, 2 \rangle$$

This vector and the original point  $(3, 1, -1)$  immediately determine the plane and line:

$$\begin{aligned} \langle 1, -2, 2 \rangle \cdot \langle x - 3, y - 1, z + 1 \rangle &= 0 \\ x - 2y + 2z &= -1 \end{aligned}$$

$$\ell(t) = (3, 1, -1) + t\langle 1, -2, 2 \rangle = (3 + t, 1 - 2t, -1 + 2t)$$

## 12.4. Videos

### 12.4.1. Calculus Blue

[https://youtu.be/Xxy\\_xvbMAew?si=sLgi-Xs2c\\_aEiEwg](https://youtu.be/Xxy_xvbMAew?si=sLgi-Xs2c_aEiEwg)

<https://youtu.be/tIuGAXwqM5M?si=GLhbf1WgWi9zHGj>

[https://youtu.be/tIuGAXwqM5M?si=WK\\_PHSDEy9YKK4uX](https://youtu.be/tIuGAXwqM5M?si=WK_PHSDEy9YKK4uX)

<https://youtu.be/SsBiqZ8JtRs?si=v4OuVbr3EM6WZ25M>

### 12.4.2. Khan Academy:

<https://youtu.be/tIpKfDc295M?si=2p92m3R1nyE9kXw4>

[https://youtu.be/\\_-02ze7tf08?si=A52-NJaNn9HU-jj](https://youtu.be/_-02ze7tf08?si=A52-NJaNn9HU-jj)

[https://youtu.be/N\\_ZRcLheNv0?si=IKKkK8H-9OthIxcB](https://youtu.be/N_ZRcLheNv0?si=IKKkK8H-9OthIxcB)

Directional derivatives and slope:

<https://youtu.be/4tdyIGIEtNU?si=tu21kvEpX00zAiJl>

Why the gradient is the direction of steepest ascent:

## 12. *The Gradient*

<https://youtu.be/TEB2z7ZIRAw?si=IvKKBQjmEqL3Ej8m>

The gradient and Contour Maps:

<https://youtu.be/ZTbTYEMvo10?si=8u9asDeBoILbVY97>

### **12.4.3. Example Problems**

[https://youtu.be/\\_qAPnUIrLqg?si=d6HSrlMLO3YezfNK](https://youtu.be/_qAPnUIrLqg?si=d6HSrlMLO3YezfNK)

<https://youtu.be/i9hhwAZ6hYs?si=5-oO4ScrIfpvLMvU>

<https://youtu.be/ErZGbQeWIAQ?si=UvQeAp2I12snMMP5>

<https://youtu.be/xBKhPZ5RgzQ?si=4Asi0yLE8RBPriFb>

<https://youtu.be/X3UjqMtWq9U?si=i8q8skju38xywdw7>

<https://youtu.be/GJODOGq7cAY?si=0O7PHxWj-HrH1C2O>

## 13. Extrema

We've developed some powerful tools for working with multivariable functions: we can take their partial derivatives, directional derivatives, and understand the relationship between the gradient and their level sets. Our next goal is to put this knowledge to work and learn how to find maximal and minimal values. This is a critical skill in real world applications, where we are looking to *maximize* efficiency, or *minimize* cost.

**Definition 13.1** (Local Extrema). An extremum is a catch-all word for a maximum or a minimum (its an *extreme value*, meaning either largest or smallest). A *local minimum* is a occurs at a point  $p = (a, b)$  if the value of the function  $f(x, y)$  is always greater than or equal to  $f(a, b)$ , when  $x$  is near  $a$  and  $y$  is near  $b$ . Analogously, a *local maximum* occurs at  $(a, b)$  if  $f(x, y) \leq f(a, b)$  for all  $(x, y)$  near  $(a, b)$ .

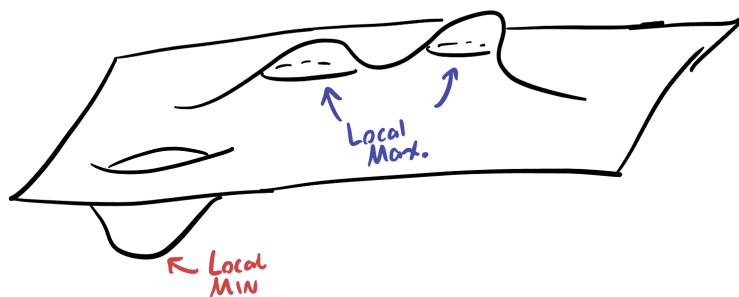


Figure 13.1.: Local Maxima and Minima of a function

In this section we will mainly be concerned with how to find local maxima and minima, though in optimization we will often be after the *maximal value* or *minimal value* of the function overall. These *global maxima* or *minima* are often just the largest or smallest of the local extrema, so our first step will be to find the local counterparts, then sort through them.

How can we find an equation to specify local extrema? In calculus I we had a nice approach using differentiation: at a local max or min a function is neither increasing

### 13. Extrema

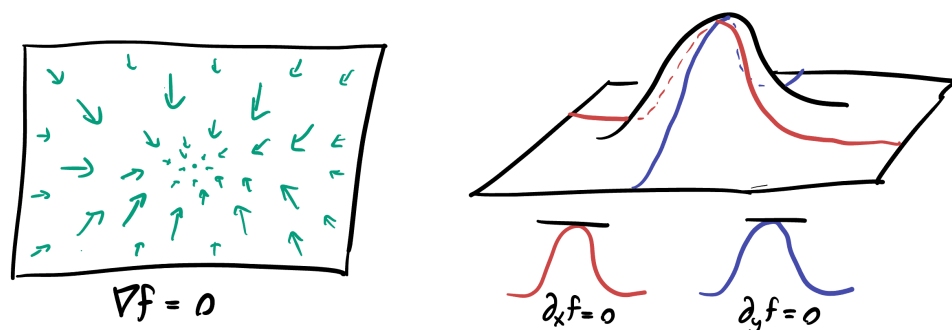


Figure 13.2.: At a local max or min, the gradient is zero.

nor decreasing so its derivative is zero. The same technique works here, where we consider each partial derivative independently!

**Theorem 13.1** (The Gradient at Extrema). *At a local max or min, every directional derivative is zero, because the point is a local max or min in every direction. In particular, all partial derivatives are zero, so the gradient is zero.*

**Definition 13.2** (Critical Points). The critical points of a function are the points where the gradient is zero.

Like in Calculus I, we have to be careful as not all critical points are actually maxima or minima. The standard example there is  $y = x^3$  which has  $y' = 3x^2$  equal to zero at  $x = 0$ , even though this is not the location of an extremum but rather a *point of inflection*. Similarly, for multivariable functions the existence of a critical point does not imply the existence of an extremum. The easiest and most common counter-example here is the *saddle*:

**Example 13.1** (Critical Points). Find the critical points of

$$f(x, y) = x^3 + y^3 + 6xy$$

Solving the system of equations arising from setting the gradient to zero is the analog of the first derivative test. What's analog of the second derivative test? Here we need to be able to take the second derivative, or to differentiate the gradient:

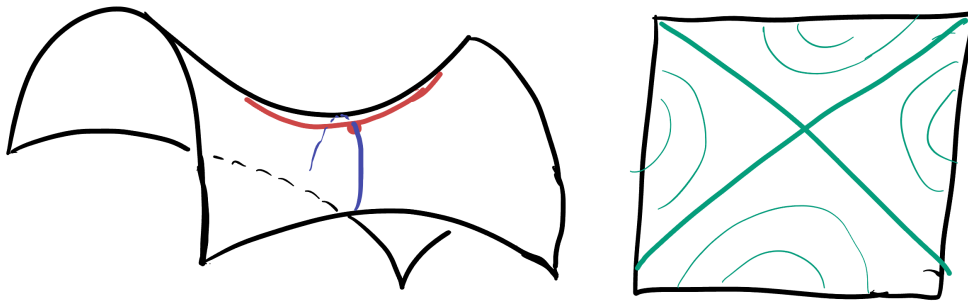


Figure 13.3.: A saddle point also has gradient zero: it's at the intersection of two contour lines - meaning there are two directions where the function has zero derivative. It's also the location of a maximum (in one direction) and a minimum (in the other) meaning the directional derivative is zero here too!

**Definition 13.3** (The Second Derivative Matrix). The second derivative of a function  $f(x, y)$  is the matrix of all four possible second order partial derivatives, much as the first derivative (the gradient) was the vector of all possible first derivatives. This matrix is often called the *hessian* and is denoted  $Hf$ :

$$Hf = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

It's helpful to recall that the order in which you take partial derivatives does not matter, so  $f_{xy} = f_{yx}$ : the off-diagonal terms in this matrix are equal to one another.

But now we have to confront the problem of how to use this information to help us find maxima and minima. As it turns out, the second derivative matrix stores the information needed to build the best possible *quadratic approximation* to our function  $f$ , just as the first order partials stored the information needed for a *linear approximation* in the gradient.

The way we read a matrix as a quadratic approximation is as follows: the matrix  $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$  represents the quadratic  $ax^2 + bxy + byx + cy^2$ . Because  $xy = yx$  we can simplify this to

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \implies ax^2 + 2bxy + cy^2$$

To understand if our function has a max, min or saddle at a given critical point, its enough to figure out if its quadratic approximation looks like a max min or saddle

### 13. Extrema

there! So, we need a way of telling whether or not the quadratic encoded by  $\begin{pmatrix} a & b \\ b & c \end{pmatrix}$  is a paraboloid (max/min) or a saddle! The important tool here is the *determinant* (which we already met when computing cross products)

$$D = \det Hf = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

Where again we've used that  $f_{xy} = f_{yx}$  to simplify our formula. The sign of this quantity determines whether the quadratic is a saddle:

**Theorem 13.2.** *If  $p$  is a critical point of  $f(x, y)$  and  $D$  is the determinant of the second derivative matrix at  $p$ , then*

- $p$  is a saddle if  $D < 0$ .
- $p$  is a minimum if  $D > 0$  and  $f_{xx} > 0$
- $p$  is a maximum if  $D > 0$  and  $f_{xx} < 0$
- Otherwise, the test is indeterminate.

It is possible to go *beyond the quadratic approximation* and understand the points this test labels indeterminate, but this requires more complicated mathematics and rarely shows up in real-world applications.

Its helpful to confirm this test is doing the right thing for examples we already understand: so let's take a quick look at a maximum, minimum and saddle:

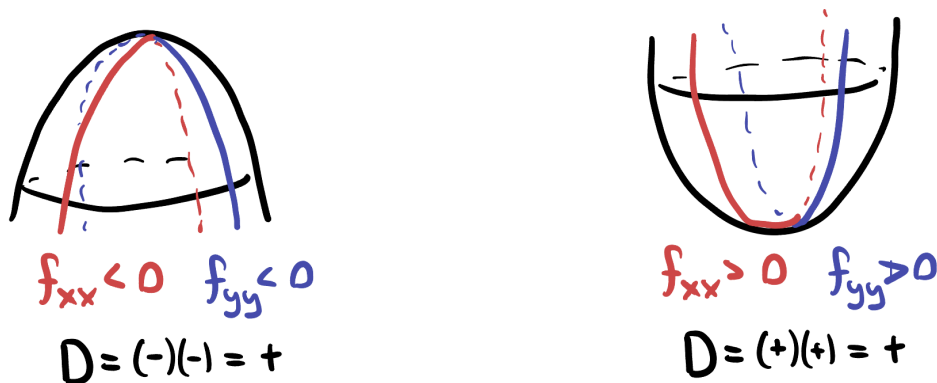
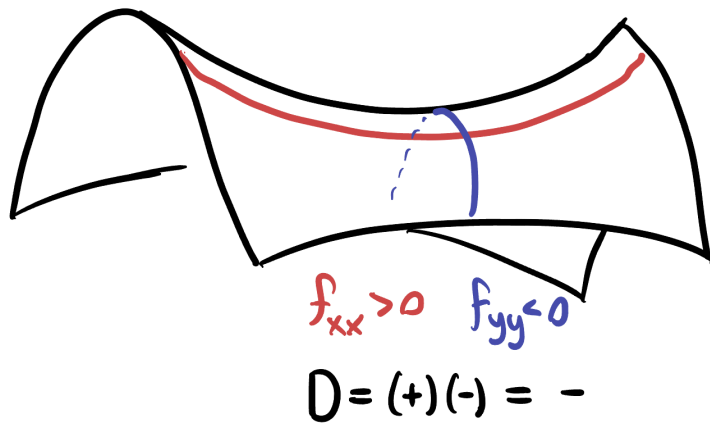


Figure 13.4.: Local maxes and mins both have  $D > 0$ .

Figure 13.5.: Saddle points have  $D < 0$ .

### 13.1. Finding Maxima Minima and Saddles

**Example 13.2**  $(x^2 + y^2 - 2x - 6y + 14)$ .

**Example 13.3**  $(x^3 + y^3 + 6xy)$ .

**Example 13.4**  $(2x^3 - yx + 6xy^2)$ .

### 13.2. Sketching Multivariate Functions

Having precise mathematical tools to understand the critical points of a function allows us to understand the total behavior of the function - because it gives us the tools to draw a contour plot! Here's an example: say we ran the above computations and found a function with three critical points: a max a min and a saddle.

We plot and label them on an  $xy$  plane, and then we can draw little local models of what the contours must look like nearby, since we know the contours for maxes mins and saddles!

Next, since there are *no other critical points* we know there isn't anything else interesting going on in our function's behavior. So, we can extend this to a drawing of the contour plot for the whole function by first extending the lines that already exist in a way that they do not cross (if they crossed anything else, that would be representing a *new saddle-*

### 13. Extrema

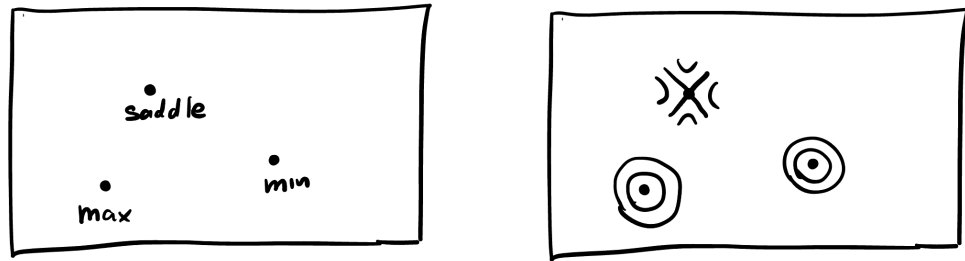


Figure 13.6.: Sketching local information from the critical points.

but we know there are none!) And then, we can just fill in contour lines in a non-intersecting way essentially uniquely, so that they create no new saddles or closed loops (which would have a new max or min in their center!)

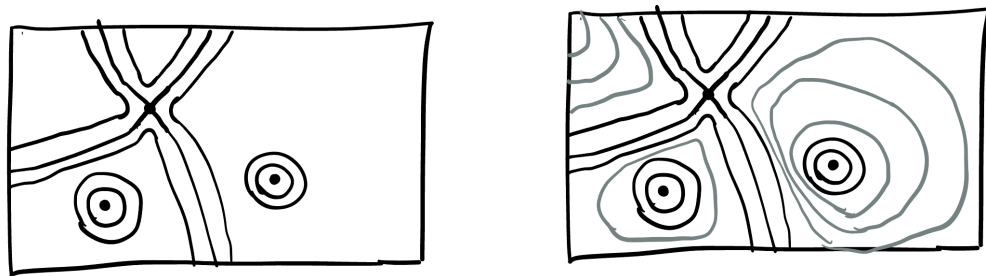


Figure 13.7.: Expanding this to an entire contour plot.

The observation that makes this possible is that nothing strange can happen away from a critical point: if the first derivative is nonzero, then the function is simply increasing or decreasing (in some direction), and the level sets nearby locally look like a set of parallel lines! This is a gateway to a huge amount of modern and advanced mathematics called *morse theory*

<https://stevejtrettel.site/code/2023/contour-slicing/>

This technique remains important even beyond scalar functions, where changes in contours signify changes in the *topology* of a shape!

<https://stevejtrettel.site/code/2023/morse-function/>



## 13.3. Videos

### 13.3.1. Calculus Blue

<https://youtu.be/J1HS6hmBtC0?si=pM8eYtifjLOADcCJ>

<https://youtu.be/XLe0YMZkh70?si=IMgHpAA-QvMQlJFB>

<https://youtu.be/7ncgs04-Epw?si=zAChu5xahRhnBm10>

[https://youtu.be/ZtmglCu\\_XE8?si=ViVsdoInIGyYNcU\\_](https://youtu.be/ZtmglCu_XE8?si=ViVsdoInIGyYNcU_)

### 13.3.2. Khan Academy

<https://youtu.be/ux7EQ3ip2DU?si=9ssdihUGBJDexoOF>

[https://youtu.be/8aAU4r\\_pUUU?si=phRaS8458TaUeHBS](https://youtu.be/8aAU4r_pUUU?si=phRaS8458TaUeHBS)

[https://youtu.be/nRJM4mY-Pq0?si=bRjP9axN\\_QsCiBF](https://youtu.be/nRJM4mY-Pq0?si=bRjP9axN_QsCiBF)

<https://youtu.be/m1FhUjMMv30?si=9GQxrx9DVAhd7kpX>

<https://youtu.be/sJo7D74PAak?si=JBmJT4Gelk2G29hb>

Example Problem:

<https://youtu.be/shWXeUn5BHk?si=4vhAu1mjaXFaf-zO>

<https://youtu.be/TqslX-bUTD8?si=IH5wv4HwsajoqUnp>

### 13.3.3. Example Problems

[https://youtu.be/RqRnKry9L3g?si=uNPUQnwhgfSand\\_I](https://youtu.be/RqRnKry9L3g?si=uNPUQnwhgfSand_I)

<https://youtu.be/YirI8OvO3tU?si=9WtKpDsp7jtDlERl>

[https://youtu.be/odkaPgWPQGo?si=-8zT4yHo5HR\\_dryb](https://youtu.be/odkaPgWPQGo?si=-8zT4yHo5HR_dryb)

[https://youtu.be/xl-4T8ak8Eg?si=ptjIrk2jzeQTIX0\\_](https://youtu.be/xl-4T8ak8Eg?si=ptjIrk2jzeQTIX0_)

*13. Extrema*

**13.3.4. Optimization Example Problems:**

[https://youtu.be/nb-\\_bs6eSYo?si=QNnkoVg3FcDY22Oc](https://youtu.be/nb-_bs6eSYo?si=QNnkoVg3FcDY22Oc)

<https://youtu.be/laKOfcVjrlc?si=cDg2tDsLthqXWZa9>

[https://youtu.be/PWAWxpG1zjg?si=1BBKqjw\\_IOubwn2T](https://youtu.be/PWAWxpG1zjg?si=1BBKqjw_IOubwn2T)

# 14. Constrained Optimization

Realistic optimization problems often involve some sort of a *constraint*:

- What is the best product we can make, with a fixed budget?
- What is the most efficient rocket we can build of a fixed mass?
- What is the least expensive building design, given the external factors of material and labor costs?

Abstractly, all of these questions ask the following: *what are the extreme values of  $f(x, y)$  given that we constrain the points  $(x, y)$  by some function,  $g(x, y) = c$ ?* In this section, we learn a couple methods to deal with such questions.

## 14.1. Method I: Reduce Dimension by Substitution

The first method is just to solve the constraint for one of the variables, and substitute it into the function you wish to optimize. This *forces* the inputs to that function to obey the constraint - so now you can just

**Example 14.1.** Maximize  $z = 4 - 2x^2 - 3y^2 + x - y$  subject to the constraint  $x + y = 2$ .

This also works in higher dimensions: we can take a problem of three variables with one constraint and turn it into a problem of two variables with no constraints:

**Example 14.2.** Find the maximum value of  $xyz$  subject to the constraint  $x + y + z = 1$ .

Unfortunately, this entire method relies on being able to *solve* the constraint for a given variable, and substitute it in! But this is often impossible. Most relations of two variables can't be solved for one or the other variable independently.

## 14.2. Method II: Lagrange Multipliers

What are we to do, if we cannot substitute the constraint in? It helps to back up and think geometrically here: in a two variable problem, we can draw the function  $z = f(x, y)$  as a surface in  $\mathbb{R}^3$ , and the constraint  $g(x, y) = c$  as a curve in the domain (the  $xy$  plane). The values of  $f(x, y)$  which satisfy the constraint

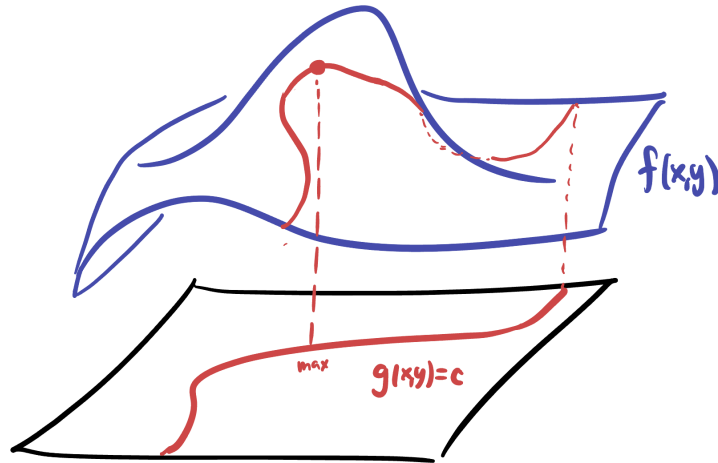


Figure 14.1.: Looking for maxima along a constraint.

It's even more helpful to draw this as a *contour plot* with level sets. The constraint (our hiker's trajectory) still appears as a curve, but we can easily read off exactly where our hiker is going uphill or downhill by looking at *how* they are crossing contours.

Whenever the hiker is crossing a contour they are either increasing or decreasing in elevation, and so cannot be at an extremum. Indeed - extrema can only occur at locations where the hiker is *not crossing a level set* - that is, where their path is tangent to a level set! This is the fundamental insight

Extrema occur when the constraint is tangent to a level set.

Everything follows from this: but the work is in turning this qualitative insight into a system of equations. The first observation we can make is that the constraint itself  $g(x, y) = c$  is a level set - just of a different function. So

Extreme values of  $f$  along the constraint  $g(x, y) = c$  occur when this constraint level set is tangent to a level set of  $f$ .

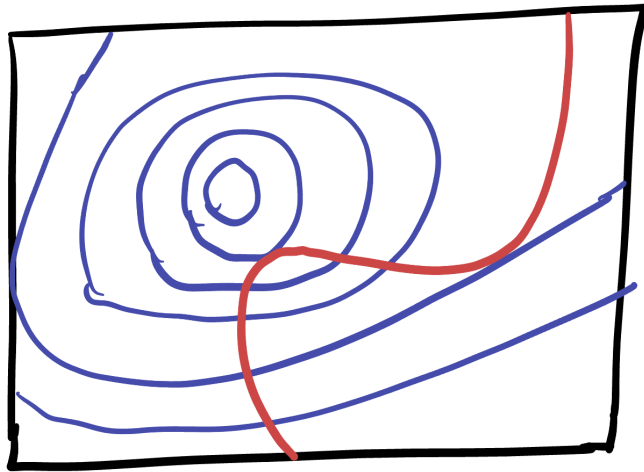


Figure 14.2.: Viewing a constraint problem with level sets.

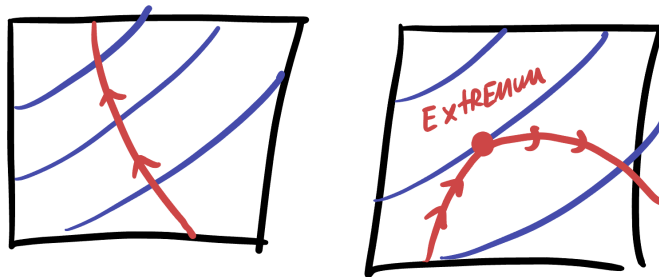


Figure 14.3.: The fundamental insight of constrained optimization.

## 14. Constrained Optimization

But this relation of *is tangent to* is still difficult to deal with. To help, we remember that the gradient vector is perpendicular to level sets! Thus,  $\nabla f$  and  $\nabla g$  are both perpendicular to their level sets, and thus these vectors must be *parallel* since the level sets are *tangent*.

Extreme values of  $f$  occur along  $g(x, y) = c$  whenever  $\nabla f$  is parallel to  $\nabla g$ .

Now we've really made some progress! We just need to remember that *parallel* means *are scalar multiples of each other* and give this scalar multiple a name: its traditional name is the greek letter  $\lambda$

Extreme values of  $f$  occur along  $g(x, y) = c$  when there exists a constant  $\lambda$  with  $\nabla f = \lambda \nabla g$ .

This is now a fully precise, quantitative claim: it tells us that we can find the constrained extrema by solving a system of equations!

$$\left\{ \begin{array}{l} \nabla f(x, y) = \lambda \nabla g(x, y) \\ g(x, y) = c \end{array} \right\}$$

Recalling that the gradient is the vector of partial derivatives, this is just the system of equations

$$\left\{ \begin{array}{l} f_x(x, y) = \lambda g_x(x, y) \\ f_y(x, y) = \lambda g_y(x, y) \\ g(x, y) = c \end{array} \right\}$$

This has three variables and three unknowns, so generically we will be able to find some finite number of solutions! Unfortunately there is no general strategy for solving such equations, other than the "substitute one into the other and think about it" technique familiar from precalculus.

It's illustrative to re-do the original example from the substitution section using the method of multipliers.  $\text{Maximize } z = 4 - 2x^2 - 3y^2 + x - y$  subject to the constraint  $x + y = 2$ .

**Example 14.3.** Maximize  $x^2 + 2y^2$  subject to the constraint  $x^2 + y^2 = 1$

**Example 14.4.** Find the maximum value of  $xyz$  subject to the constraint  $x + y + z = 1$ .

### 14.3. Optimization and Inequalities:

For dealing with an inequality, we need to break the problem into two cases: when the constraint is an equality, we can do the same process we have been learning above, with either Lagrange multipliers or substitution. And, inside of the constraint, we can do our more standard two dimensional optimization (finding critical points, sorting into maxes and mins), and be careful only to consider critical points that are *inside the domain we care about*: if the constraint is  $x^2 + y^2 < 1$  and you find a critical point  $(3, 0)$  you can ignore it, but the critical point  $(1/2, 1/2)$  needs to be considered.

This will result in you having two sets of potential extrema: those occurring on the inside, and those occurring on the boundary. How do you find the absolute max (or min)? That's easy! Just take the largest (or smallest) overall result.

**Example 14.5.** Maximize  $x^2 + 2y^2$  subject to the constraint  $x^2 + y^2 \leq 1$





**Part IV.**

# **Multivariate Integration**



## 15. Double Integrals

We now begin a new chapter - after studying in detail various means of studying *change* via multivariate differentiation, we will switch to study *accumulation* via multivariate integration. As Calculus I and II focused on defining the integral of a single variable over a 1-dimensional region (the closed interval  $[a, b]$ ), we will continue in Calculus III to define the integral of multivariate functions over two and three dimensional regions.

### 15.1. Riemann Sums and Iterated Integrals

In one dimension, an integral measures the area under a graph by breaking in into slices, and adding up approximate areas of each slice, via a Riemann sum, before taking a limit. We will begin with a similar process here, we define the *double integral* of a function  $f(x, y)$  over a region  $R$  in the plane by a two dimensional Riemann sum.

<https://stevejtrettel.site/code/2022/riemann-sum-2d/>

This two dimensional Riemann sum works by breaking the region  $R$  into small rectangular regions which we will denote  $\Delta A$ , choosing a point  $(x_i, y_j)$  in each such region, and then summing

$$\sum_{i=1}^N \sum_{j=1}^N f(x_i, y_j) \Delta A$$

As the number of regions goes to infinity, and the size of each rectangle  $\Delta A$  goes to zero, this becomes an *integral*, with  $\Sigma$  becoming  $\int$  and  $\Delta$  becoming  $d$ :

$$\iint_R f(x, y) dA$$

This measures the *volume under the graph* of  $f$  above the region  $R$ , instead of the area under a curve. But how do we evaluate this thing? We can either add up the volume of each row with constant  $x$  first, to get a function of  $y$ , and then add these up, or the opposite: first add up in rows of  $y$  to get a function of  $x$ , then add these up. Either way, we add up all the little volumes, and this gives the total volume under the surface.

## 15. Double Integrals

You can see this in the animation below: where one of the side bar graphs gives the result of summing along rows first, the other columns, and these two side graphs have the same total area under their curves.

<https://stevejtrettel.site/code/2022/fubini>

### 15.2. Rectangular Domains

Let  $R$  be the region  $a \leq x \leq b$  and  $c \leq y \leq d$ . Say we want to compute the integral  $\iint_R f(x, y) dA$ . By the observation above (Fubini's theorem) we can compute this by integrating all the  $x$ 's first then integrating  $y$ , or vice versa:

$$\iint_R f(x, y) dA = \int_a^b \left( \int_c^d f(x, y) dy \right) dx = \int_c^d \left( \int_a^b f(x, y) dx \right) dy$$

<https://stevejtrettel.site/code/2023/iterated-integral-cartesian>

This is a massive simplification: it means that we can compute two dimensional integrals by just doing two one dimensional integrals, one after the other!

**Example 15.1.** Evaluate  $\iint_R x^2 y dA$  for  $R = [1, 2] \times [3, 4]$

**Example 15.2.** Evaluate  $\iint_R x(3 - y^2) dA$  for  $R = [0, 2] \times [1, 2]$

Oftentimes, the order one performs the integrals in does not matter - both are equally straightforward. But this is not always the case!

**Example 15.3.** Integrate  $y \sin(xy)$  over the region  $R = \{0 \leq x \leq 1, 0 \leq y \leq \pi\}$ .

*Try both orders, see which is easier!*

Sometimes, when the function you are integrating is a product of a function of  $x$  and a separate function of  $y$ , things can simplify even further! If  $f(x, y) = g(x)h(y)$  then we may write

$$\iint_R = \int_a^b \int_c^d g(x)h(y) dy dx = \left( \int_a^b g(x) dx \right) \left( \int_c^d h(y) dy \right)$$

We get this by realizing that  $g(x)$  is a constant with respect to the  $y$  integral so we can pull it out: but then once we have done the  $y$  integral the result is a constant, so we can pull it out of the  $x$  integral!

**Example 15.4.** Compute

$$\iint_R e^x \sin(y) dA$$

On the region  $R = [0, \pi/2] \times [0, \pi/2]$ .

This is essentially all there is to the theory of multiple integrals when the domain is a box (where all variables are bounded by constants). Indeed, we will shortly meet triple integrals and see that everything remains precisely the same!

$$\int_R g(x, y, z) dV = \int_a^b \left( \int_c^d \left( \int_e^f g(x, y, z) dz \right) dy \right) dx$$

However, before going there we will continue on and look at more general double integrals: what happens when the region  $R$  is not a box?

### 15.3. Variable Boundaries

In one variable calculus, the only sort of region over which you could perform an integral is a single interval. But in two variables, the regions of the plane over which you could wish to integrate are much more varied!



We have learned how to deal with rectangular regions by *slicing* - and this same technique will serve us well in many other cases. To start, we won't focus on completely general regions, but rather on regions where the top and bottom are bounded by functions of  $x$ :

Here, if we slice with respect to  $y$  first, our vertical slices will each be of a different length - but they will still always be intervals (the top and bottom are *functions* meaning they pass the vertical line test - so each intersects the vertical strip exactly once).

At the fixed value  $x$ , what is the interval we are integrating over? Well, it runs from the bottom function  $a(x)$  to the top function  $b(x)$ , and so the integral along this slice is

15. Double Integrals

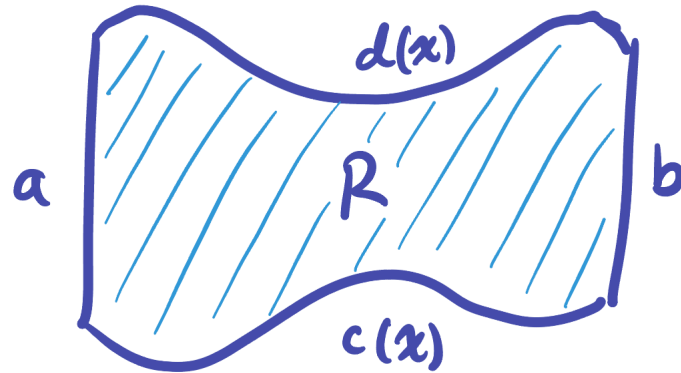


Figure 15.1.: A region where the top and bottom boundaries are functions of  $x$

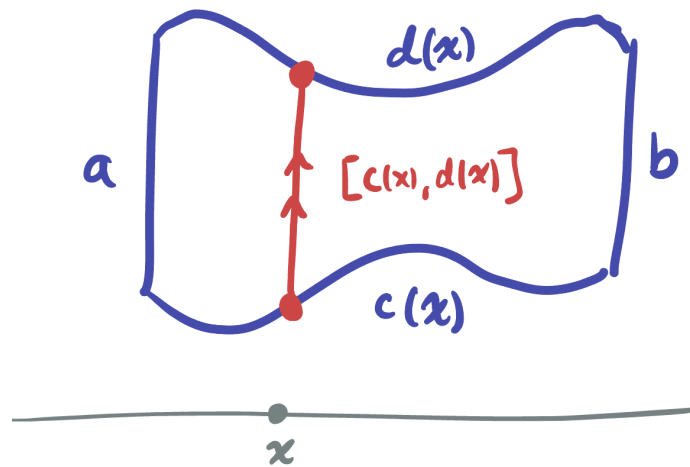


Figure 15.2.: Vertical slices of such a region are intervals.

$$\int_{c(x)}^{d(x)} f(x, y) dy$$

Then all that remains is to integrate this along the  $y$  direction:

$$\iint_R f(x, y) dA = \int_a^b \int_{c(x)}^{d(x)} f(x, y) dy dx$$

<https://stevejrettel.site/code/2023/iterated-integral-variable-bounds>

**Exercise 15.1.** Find the integral of  $x + 2y$  over the region  $R$

$$R = \{(x, y) \mid 0 < x < 2 \quad x^2 - 2 < y < x\}$$

Sometimes the  $x$  bounds don't even need to be given explicitly—they are just the region between where the curves intersect:

**Exercise 15.2.** Find the volume above the  $xy$  plane under the graph of  $z = x^2 + y^2$ , within the region  $R$  bounded by  $y = 2x$  and  $y = x^2$ .

There's nothing special about slicing with respect to the  $x$  direction, we can also do integrals by slicing with respect to fixed  $y$ , and integrating  $dx$  first. Indeed, the above example can be redone this way no problem!

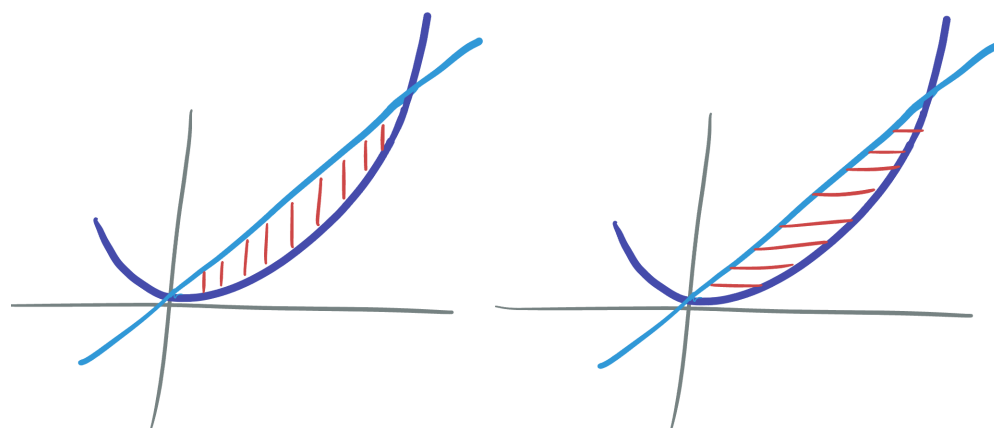


Figure 15.3.: Slicing the same region either vertically or horizontally.

## 15. Double Integrals

**Exercise 15.3.** Find the volume above the  $xy$  plane under the graph of  $z = x^2 + y^2$ , within the region  $R$  bounded by  $y = 2x$  and  $y = x^2$ , this time first slicing horizontally, with constant  $y$ .

However, not every example is just as easy both ways. For example the following integral is easy to write down sliced with respect to  $y$ , but harder when sliced first with constant  $x$ :

**Exercise 15.4.** Integrate  $x + y$  on the region  $R$  determined by

$$R = \{(x, y) \mid -2 < y < 2 \quad y^2 - 1 < x < 3\}$$

### 15.3.1. Changing the Order of Integration

To do the last integral instead with respect to slices of constant  $x$  (so, slices in the  $y$  direction) we would need to solve for the  $y$  bounds as a function of  $x$ ,

**Exercise 15.5.** Set up the integral of  $x + y$  on the region

$$R = \{(x, y) \mid -2 < y < 2 \quad y^2 - 1 < x < 3\}$$

Where slicing is first done at constant  $x$ s.

One of the most important things about setting up a double integral correctly is thinking through which order of integration will be more useful, and why. Sometimes, one way of slicing will lead to an impossible integral, but the other way will be easy!

**Example 15.5.** Compute the integral

$$\int_0^1 \int_x^1 \sin(y^2) dy dx$$

by switching the order of integration first.



## 15.4. Combining Integrals

Just like there is a *subdivision rule* for one dimensional integrals,

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx$$

There is a similar rule for double integrals: if you break the domain into two regions, the double integral over the whole thing is the sum of the double integrals over each. In symbols: if  $R = R_1 \cup R_2$ , then

$$\iint_R f dA = \iint_{R_1} f dA + \iint_{R_2} f dA$$

This lets us perform integrals we otherwise could not, by breaking the domain down into simpler pieces, which we can then slice with respect to either  $x$  or  $y$ .



## 16. Triple Integrals

Triple integrals follow a very similar general theory to double integrals: starting with a function  $f(x, y, z)$  on  $\mathbb{R}^3$ , we define the integral over a region  $E$  by breaking that region into small cubical volumes of size  $dV$  and building a 3-dimensional riemann sum. Taking the limit gives the triple integral, or

$$\iiint_E f dV$$

To evaluate such an expression, we need to break the integral into slices, and evaluate them one at a time. Such slicing relies on understanding the volume element in three dimensions, which is the volume of an infinitesimal box

$$dV = dx dy dz$$

This lets us separate the triple integral into three consecutive integrals: first  $dx$ , then  $dy$  then  $dz$ . Or, because the order of multiplication doesn't matter, we could do the integral in any of the other six possible orders

$$\begin{aligned} dx dy dz &= dx dz dy = dy dx dz \\ &= dy dz dx = dz dx dy = dz dy dx \end{aligned}$$

### 16.1. Different Bounds:

#### 16.1.1. Boxes

When the domain  $E \subset \mathbb{R}^3$  is a coordinate box, described as

$$E = \{(x, y, z) \mid a \leq x \leq b, c \leq y \leq d, e \leq z \leq f\}$$

This triple integral splits into an iterated integral with constant bounds:

## 16. Triple Integrals

$$\iiint_E f dV = \int_a^b \int_c^d \int_e^f f dz dy dx$$

This integral could be done in any of the six possible orders, as all the bounds are constants, no order will be easier or harder than any other.

### 16.1.2. Variables in One Bound

If the domain  $E$  is described so that its bounds in at two of the variables are constants, and the third set of bounds are variables, then there is a preferred order in which to integrate. In particular, we know the final answer must be a number so we cannot have variables in the outermost set of bounds, and must be done earlier: the easiest situation is just to do it first!

For example, consider the following domain  $E$ :

$$E = \{(x, y, z) \mid 0 \leq x \leq 2, 0 \leq y \leq 3, 0 \leq z \leq x + y\}$$

Here the  $z$  bound is different depending on which point  $(x, y)$  you are at, so we do the  $z$ -integral first. Since both the  $x$  and  $y$  bounds are constants

$$\iiint_E f dV = \int_0^2 \int_0^3 \int_0^{x+y} f dz dy dx$$

### 16.1.3. Variables in Two Bounds

For more complicated domains, its possible that variables will appear in two of the bounds. (Because the final answer must be a number, we know the outer bounds must be constants, so they cannot appear in all three bounds).

In such cases, the innermost integral can have bounds depending on two variables (the next two to be integrated), and the middle integral can have bounds depending on the outermost integral. This way, at each stage the function only has variables left in it that are still going to be integrated away, and the result is a number. In this case, there is only *one* possible order in which the integral can be performed!

Here's an example: if  $E$  is the following region

$$E = \{-y \leq x \leq yz, 0 \leq y \leq z + 1, -1 \leq z \leq 1\}$$

## 16.2. Describing the Bounds:

Then a triple integral must be performed with  $dx$  first, then  $dy$ , and finally  $dz$ :

$$\iiint_E f \, dV = \int_{-1}^1 \int_0^{z+1} \int_{-y}^{yz} f \, dx \, dy \, dz$$

## 16.2. Describing the Bounds:

The thing that makes triple integrals challenging is not doing the integrals (its just three 1D integrals) or even choosing the order to do them in (as we saw above, once you have described the domain in terms of  $x$ ,  $y$ ,  $z$ , its easy to decide which order to do the integral.) The difficult part is often just describing the bounds themselves!

This is mostly because visualizing 3D geometry takes some training to get used to! It's helpful to look through many examples: please remember to be using the book (**chapter 15**), where each chapter is essentially just a giant list of example problems fully worked out! Additionally, Here is another collection of fully worked examples online:.

## 16.3. Video Resources



## 17. Integrals & Coordinates

We've seen previously that certain double and triple integrals are particularly challenging because their bounds contain complicated expressions like  $\sqrt{1-x^2}$ , which lead to you having to do an *integral* of functions containing things like  $\sqrt{1-x^2}$  which leads to difficult trigonometric substitutions, or worse.

These sort of expressions come up when integrating over circular, cylindrical and spherical regions, because these are all described with equations like  $x^2 + y^2 = 1$  or  $x^2 + y^2 + z^2 = 1$  in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ . And this chapter is the bearer of good news: the reason these integrals look hard at first is that the cartesian coordinates  $x, y, z$  are not a good way to work with them. But, after adapting our viewpoint, all the square roots melt away and these integrals become straightforward to compute!

The reason is that cartesian coordinates are good for describing flat objects: the surfaces where one variable is held constant describe lines or planes. Thus, integrals over rectangles and boxes are easy in cartesian coordinates: their bounds are constants! To make integrals over circles, cylinders and spheres easy, we need to find coordinates for which circles, cylinders and spheres are described by constants. If we can change our perspective to work with these coordinates, we will be able to turn an integral with difficult bounds into a different integral with constant bounds - but the same overall value.

### 17.1. Polar Coordinates

Polar coordinates are a means of representing the plane using distance  $r$  from the origin, and angle  $\theta$  from the  $x$ -axis.

Using trigonometry, we can relate these to the usual  $x$  and  $y$  coordinates we are used to.

**Definition 17.1** (Polar Coordinates). Polar coordinates on the plane are the coordinates  $r, \theta$  where  $r$  measures the distance from the origin, and  $\theta$  measures the angle from the  $x$ -axis. The conversion from cartesian to polar coordinates is given by the functions

## 17. Integrals & Coordinates

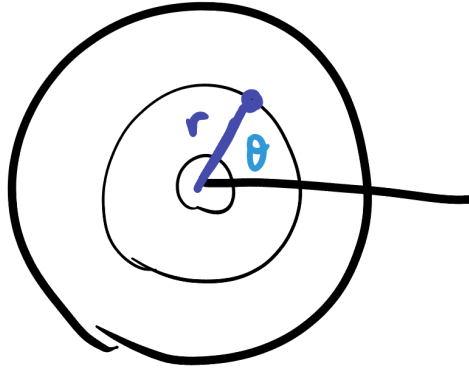


Figure 17.1.: Polar Coordinates definition.

$$x = r \cos \theta \quad y = r \sin \theta$$

**Definition 17.2** ( $dA$  in Polar Coordinates).

$$dA = r dr d\theta$$

This lets us do a double integral in polar coordinates by first doing an  $r$  integral, and then a  $\theta$  integral or vice-versa:

$$\iint_R f dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f r dr d\theta$$

Just like in cartesian coordinates, you can view this as slicing in the  $r$  and  $\theta$  directions, and integrating the results.

<https://stevejtrettel.site/code/2023/iterated-integral-polar>

Starting from an integral with cartesian coordinates  $x, y$ , there is a straightforward procedure to convert to polar:

- Convert the function to polar coordinates: substitute  $x = r \cos \theta$  and  $y = r \sin \theta$  and simplify (remember  $x^2 + y^2 = r^2$ ).
- Substitute  $dA$  or  $dx dy$  for the polar area unit  $dA = r dr d\theta$ .
- Rewrite the bounds of integration in terms of polar coordinates.

Now you just have a standard iterated integral (but with variables named  $r$  and  $\theta$  instead of  $x$  and  $y$ .) This can be computed as normal: just doing one integral at a time.



**Example 17.1.**

## 17.2. Cylindrical Coordinates

Cylindrical coordinates are just the natural three dimensional extension of polar coordinates, where we use  $r$ ,  $\theta$ , and  $z$ .

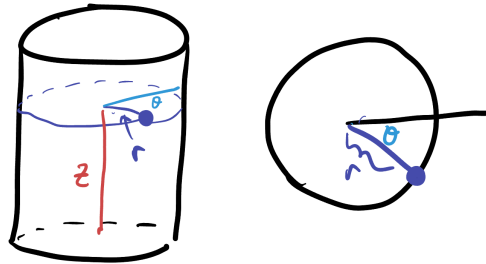


Figure 17.2.: Cylindrical coordinates definition

**Definition 17.3** (Cylindrical Coordinates). Measure two directions in space using polar coordinates, and the orthogonal direction with its standard Cartesian axis. If we convert the  $xy$  plane to polar, this means

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

The volume element here is just the polar area element times  $dz$ :

**Definition 17.4** (Volume in Cylindrical Coordinates).

$$dV = (dA)dz = r dr d\theta dz$$

I've recorded example videos of this, for our asynchronous lecture day! The video links are available on canvas.

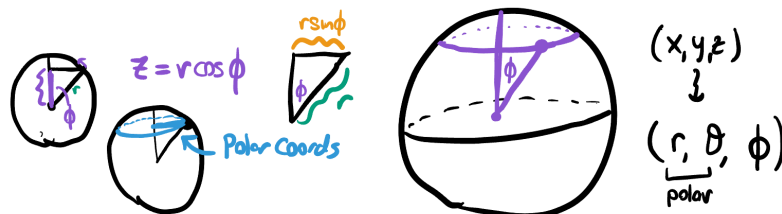


Figure 17.3.: Defining spherical coordinates.

### 17.3. Spherical Coordinates

Spherical coordinates is a coordinate system in  $\mathbb{R}^3$  where we represent a point with latitude, longitude, and radius.

**Definition 17.5** (Spherical Coordinates).

$$x = r \cos \theta \sin \phi$$

$$y = r \sin \theta \sin \phi$$

$$z = r \sin \phi$$

Using these coordinate definitions we can compute the volume element in spherical coordinates: it'll be a product of the length in the  $r$  direction, the length in the  $\theta$  direction and the length in the  $\phi$  direction.

- Length in the  $r$  direction is  $dr$ .
- Circles in the  $\theta$  direction (lines of longitude) have circumference  $2\pi r \sin \phi$ . Thus a small amount of angle has length  $r \sin \phi d\theta$ .
- Circles in the  $\phi$  direction are all longitudes on the sphere, of length  $2\pi r$ . Thus a small bit of angle has length  $rd\phi$ .

**Definition 17.6** (Volume in Spherical Coordinates).

$$dV = dr(r \sin \phi d\theta)(rd\phi)$$

$$= r^2 \sin \phi dr d\theta d\phi$$

For examples of doing integrals in spherical coordinates, I recorded videos and posted them to our course Canvas page for the asynchronous lecture.

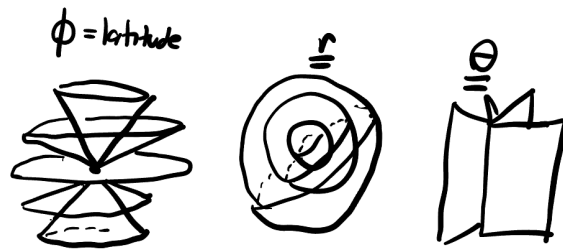


Figure 17.4.: Surfaces of  $r, \theta, \phi = \text{constant}$ .

## 17.4. Video Resources



**Part V.**

# **Homework**



# Assignment 1

This assignment concerns the topics covered in *Dimensions, Vectors and Operations*

## Dimensions

This question has **two parts**. Be sure to answer both.

Suppose you are trying to train an AI algorithm for the United States Mail service, that takes in a picture of a box being shipped, and estimates from that photograph the length, width and height of the box. That way, the computer software can quickly take a picture of a stack of boxes, and design an algorithm for how to most efficiently stack them on the truck.

**Exercise 17.1. Part 1:** Assume that the input photos are taken in black and white (so, each pixel needs only one color to specify) and of size 100 pixels wide and 75 pixels tall. As mathematics students, we realize this AI is really just performing a high dimensional function: it is taking as input one of these images, and returning its estimate of length, width and height as its output. What is the dimension of the domain (the inputs), and what is the dimension of the range (the outputs)?

**Part 2:** What would these dimensions change to if in the future you are asked to update the software to take HD color images (with red green and blue specified for every pixel on a  $1920 \times 1080$  computer screen), and the program was to estimate not just the dimensions of the box, but also its weight, and the length of time it will take to reach its destination?

## Vector Addition

This question has **two parts**. Make sure to answer both.

You are in charge of controlling a robotic arm which has five joints, and so its configuration in space can be modeled by an 5-tuple of numbers, one specifying the the

### Assignment 1

configuration of each joint. That is, the first number tells the first joint how much to move, the second number tells the second joint how to move, etc.

**Exercise 17.2. Part 1:** Starting from the “off” position, which corresponds to the point  $(0, 0, 0, 0, 0)$ , the robot is programmed to do the following sequence of events:

- Move Motor One by 1 unit.
- Move Motor Two by 2 units.
- Leave Motor 3 fixed.
- Move Motor 4 backwards 1 unit.
- Leave Motor 5 fixed

**What is the vector that represents this command to the robot?**

**Part 2:** After this, the robot is programmed to move a along the vector in direction  $\langle 1, 1, 1, 1, 0 \rangle$  of length 4. This leaves the robot arm at some new configuration in space. At this point, you wish to return the robot arm to its starting location  $(0, 0, 0, 0, 0)$ . **What direction vector should you command the robot to follow so that it does so?**

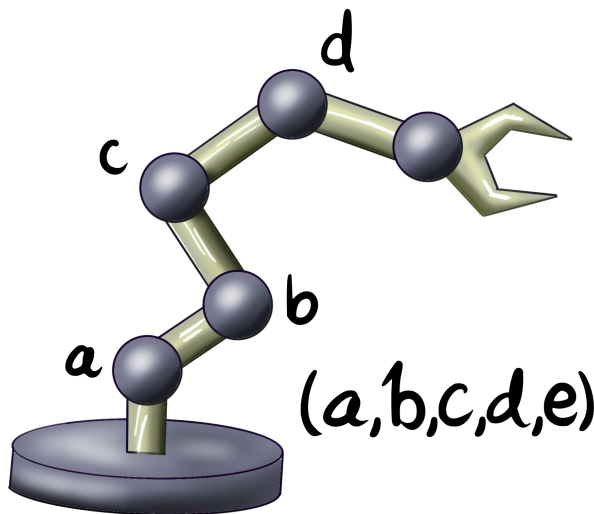


Figure 17.5.: A robot arm with a 5 dimensional configuration space.

When you write your answer, show all your work and *explain* what you are doing in full sentences where necessary to make it so a classmate could easily follow your solution.



## Diagonal of a Cube

This question has **three parts**. Make sure to answer all three.

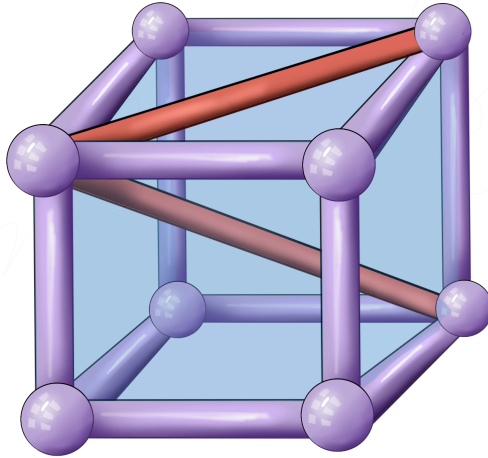


Figure 17.6.: A cube, its face diagonal, and main diagonal, in 3 dimensional space.

**Exercise 17.3. Part 1:** What angle does the diagonal of a cube make with one of its faces? To calculate this, we can find the angle between the vector  $d$  which represents the diagonal of the entire cube, and the vector  $f$  which represents the diagonal of one of the faces. Use the geometry of dot products to get  $\cos \theta$  by hand, and then use a calculator to find  $\theta$ ? (Make sure you state if your answer is in degrees or radians).

**Part 2:** Why does the side length of the cube not matter in the calculation above?

**Part 3:** What is the angle between the diagonal of a 4-dimensional cube and one of its 3-dimensional faces?

*Hint: choose a size for the cube, and assign coordinates to its vertices. Then use these coordinates to determine the vectors corresponding to each of the diagonals.*

When you write your answer, show all your work and *explain* what you are doing in full sentences where necessary to make it so a classmate could easily follow your solution.

## Solutions

### Dimensions

**Question 1:** The domain is a space of images. If photos are  $100 \times 75$  pixels, there are 7,500 pixels in an image. If the image is black and white we need one real number to specify each pixel, so we need 7,500 numbers total to specify the image: thus, the domain is 7,500-dimensional, or  $\mathbb{R}^{7500}$ . The range is three dimensional or  $\mathbb{R}^3$ , as we want the computer to output for each image three real numbers: a length (1), width (2) and height (3). Thus, our function is a map

$$AI : \mathbb{R}^{7500} \rightarrow \mathbb{R}^3$$

**Question 2:** Color images require three numbers per pixel, so the dimension of the space is going to be three times the number of pixels. An HD image has  $1920 \times 1080 = 2,073,600$  pixels, so the domain will be  $3 \times 2,073,600 = 6,220,800$  dimensional. The output will now be a point in  $\mathbb{R}^5$  as we want to predict two additional numbers, for a total of five: length width height weight and delivery time. Thus the new function is

$$AI : \mathbb{R}^{6,220,800} \rightarrow \mathbb{R}^5$$

### Vector Addition

Following the procedure, we start at  $(0, 0, 0, 0, 0)$  to give the command that corresponds to the sequence of steps we want the robot to follow, we make a vector that has in the  $n^{th}$  slot what we want the robot to do: here, that is

$$\langle 1, 2, 0, -1, 0 \rangle$$

Performing this command, we add this to the original vector, which is  $(0, 0, 0, 0, 0)$  and we reach the configuration  $(1, 2, 0, -1, 0)$ .

Next, we are asked to move the robot in the direction  $v = \langle 1, 1, 1, 1, 0 \rangle$  by a vector of length 4. To do so, we first need to find the length of  $v$ :

$$\|v\| = \sqrt{1^2 + 1^2 + 1^2 + 1^2 + 0^2} = \sqrt{4} = 2$$

To make this vector length 4, we need to scalar-multiply it by 2, to double its length. This gives us the vector

$$2v = \langle 2, 2, 2, 2, 0 \rangle$$

Adding this to our current robot configuration gives the resulting one:

$$(1, 2, 0, -1, 0) + \langle 2, 2, 2, 2, 0 \rangle = (3, 4, 2, 1, 0)$$

Finally, we are asked to figure out what command to give the robot to get from here back to its starting configuration  $(0, 0, 0, 0, 0)$ . Remember, if you want to find a vector from a point  $p$  to a point  $q$ , you need to subtract the starting point from the ending point:  $u = q - p$ . In our case, this means we want to subtract where we are at  $(3, 4, 2, 1, 0)$  from where we want to end up  $(0, 0, 0, 0, 0)$  to get

$$(0, 0, 0, 0, 0) - (3, 4, 2, 1, 0) = \langle -3, -4, -2, -1, 0 \rangle$$

Which is the final command we want to send the robot.

## Diagonal of a Cube

**Question 1:** Let's choose our cube to be a unit cube, and let's put one of its vertices at  $(0, 0, 0)$ . This means it also has vertices one unit along the  $x$ ,  $y$ , and  $z$  axes at  $(1, 0, 0)$ ,  $(0, 1, 0)$ ,  $(0, 0, 1)$ . Filling in the rest of the vertices we have:

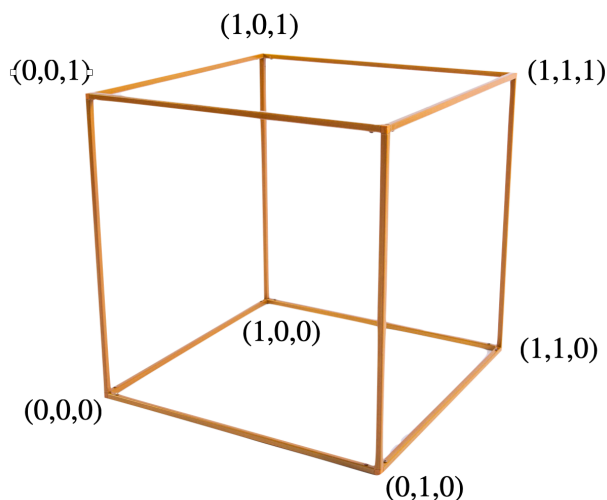


Figure 17.7.: One way of labeling all the cube's vertices. We pick one to be  $(0, 0, 0)$  and then its neighbors are  $(1, 0, 0)$ ,  $(0, 1, 0)$  and  $(0, 0, 1)$ . The remaining labels are determined by vector addition!

### Assignment 1

We will write both of our diagonals as originating at the upper left corner, which is labeled  $(0, 0, 1)$  in the diagram above. This means we want to compute the face diagonal by subtracting this from its other endpoint, the far vertex  $(1, 1, 1)$ :

$$f = (1, 1, 1) - (0, 0, 1) = \langle 1, 1, 0 \rangle$$

Similarly, we compute the main diagonal by subtracting the starting vertex from its end  $(1, 1, 0)$ :

$$d = (1, 1, 0) - (0, 0, 1) = \langle 1, 1, -1 \rangle$$

We've succeeded in turning the entire problem into vectors, so now we can answer it with vector tools! The angle  $\theta$  between  $f$  and  $d$  is defined by the dot product

$$\cos \theta = \frac{f \cdot d}{\|f\| \|d\|}$$

Computing what we need:

$$f \cdot d = \langle 1, 1, 0 \rangle \cdot \langle 1, 1, -1 \rangle = 1 + 1 + 0 = 2$$

$$\|f\| = \sqrt{1^2 + 1^2 + 0^2} = \sqrt{2}$$

$$\|d\| = \sqrt{1^2 + 1^2 + (-1)^2} = \sqrt{3}$$

Thus,

$$\cos \theta = \frac{2}{\sqrt{2}\sqrt{3}} \implies \theta = \arccos\left(\frac{\sqrt{2}}{\sqrt{3}}\right) \approx 0.61547\text{rad}$$

**Question 2:** The side length of the cube didn't matter as if we had made the cube side length  $s$ , then all of the vectors above would have been multiplied by  $s$ . This would have made both  $\|f\|$  and  $\|d\|$  longer by a factor of  $s$ , so the denominator would have grown by a factor of  $s^2$ . But since *both* of  $f$  and  $d$  were multiplied by  $s$ , the dot product  $f \cdot d$  would also have grown by a factor of  $s^2$ , and this would have cancelled the denominator! Thus, no matter how long the cube sides are,  $\frac{f \cdot d}{\|f\| \|d\|}$  would be the same number, so  $\theta$  would be the same.

**Question 3:** To run this argument in four dimensions, we can no longer picture what is going on directly but need to extrapolate. We took the "base point" in our picture to be  $(0, 0, 1)$  in 3D, so perhaps we take  $(0, 0, 0, 1)$  as the base point in  $\mathbb{R}^4$ . Making the analogs of  $d$  and  $f$  we find

$$f = (1, 1, 1, 1) - (0, 0, 0, 1) = \langle 1, 1, 1, 0 \rangle$$

$$d = (1, 1, 1, 0) - (0, 0, 0, 1) = \langle 1, 1, 1, -1 \rangle$$

Running the same computation as above we find

$$f \cdot d = 3 \quad \|f\| = \sqrt{3} \quad \|d\| = \sqrt{4} = 2$$

$$\cos \theta = \frac{f \cdot d}{\|d\| \|f\|} = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

Unlike in 3D this is one of the angles we know from the unit circle, so we can compute  $\theta$  exactly!

$$\theta = \frac{\pi}{6} \approx 0.52359\text{rad}$$



# Assignment 2

This assignment concerns the topics covered in *Operations* and *Shapes*

## Dot Products and Image Processing

You are writing software to deal with images on a smartphone, and hope to automatically sort similar photos into groups to make things more convenient for the user. Internally to the phone, assume that a photo is stored as an ordered array of numbers representing the color of each pixel (as three numbers, one for each of red, green and blue). The picture below shows how a very simple  $2 \times 2$  pixel image is represented as a list of 12 numbers, which is a vector in  $\mathbb{R}^{12}$ .

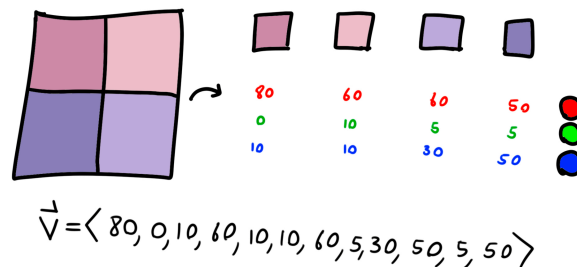


Figure 17.8.: Pixels in an image as components of a vector.

While it is very difficult to teach a computer how to understand the *content* of a photograph from the values of its pixels, it is easy to perform *vector operations* on a photograph since they are stored as arrays of numbers. In particular, it is possible to measure the *magnitude* of a photograph stored as the vector  $\langle p_1, p_2, p_3, p_4, \dots, p_n \rangle$  using the  $n$ -dimensional Pythagorean theorem, and the dot product between this image and another  $\langle q_1, q_2, \dots, q_n \rangle$  is computed as  $\vec{p} \cdot \vec{q} = \sum_{i=1}^n p_i q_i$ .

Say that your software receives three new photographs taken by the phone, call them photos  $A = \langle a_1, \dots, a_n \rangle$ ,  $B = \langle b_i \rangle$  and  $C = \langle c_i \rangle$ . The computer immediately computes the

## Assignment 2

following quantities:

$$|A| = 1050 \quad |B| = 2040 \quad |C| = 3012$$

$$A \cdot B = 1,934,567 \quad B \cdot C = 143 \quad A \cdot C = -192,456$$

**Exercise 17.4.** Which two photos do you think are of the same object? Why? (*Hint: can you figure out the 'angle' between these images in high dimensional space?*). When you write your answer, show your work and *explain* what you are doing in full sentences where necessary so a classmate could easily follow your solution.

## Planes and Perpendicularity

**Exercise 17.5.** In three dimensions, any two planes that are not parallel intersect each other in a line. If  $3x + 2y - z = 4$  and  $x - z = 16$  are two such planes, find a vector  $\vec{v}$  that is parallel to the line they intersect in.

*Hint: can you find a vector that is perpendicular to each plane? How can you use these to find a third vector which is parallel to **both** planes?*

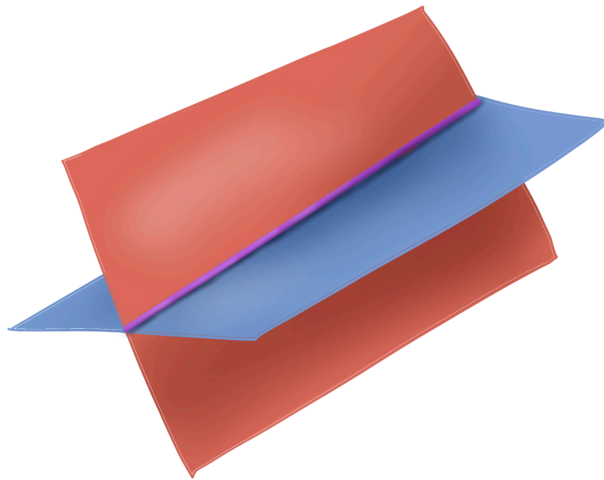


Figure 17.9.: Two planes intersecting along a line in 3 dimensional space.

When you write your answer, show all your work and *explain* what you are doing in full sentences where necessary to make it so a classmate could easily follow your solution.



## Shapes

**Exercise 17.6.** The plane  $y = 3$  intersects the sphere  $x^2 + y^2 + z^2 - 2x - 4y + 1 = 0$  in a circle. What's the radius of this circle?

When you write your answer, show all your work and *explain* what you are doing in full sentences where necessary to make it so a classmate could easily follow your solution. You should not just tell me the number, or give me a few lines of unjustified algebra!

*Hint: If all points on the plane have their  $y$  coordinate equal to 3, then every point on the circle we are interested in also has  $y = 3$ ...can you use this to get an equation only in terms of  $x$  and  $z$ ?*

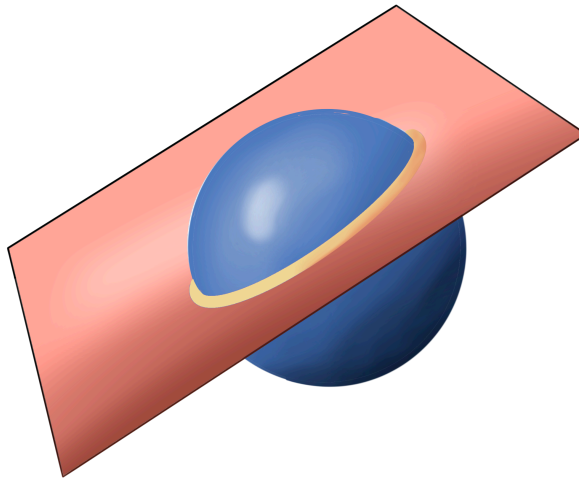


Figure 17.10.: A sphere intersects a plane in a circle.

## Solutions

### Dot Products and Image Processing

Using the relationship of dot products and angles, we can figure out what the angle between each pair of images are. To label them, I'll write  $\theta_{AB}$  for the angle between  $A$  and  $B$ , and so on:

$$\cos \theta_{AB} = \frac{A \cdot B}{\|A\| \|B\|}$$

## Assignment 2

$$\cos \theta_{BC} = \frac{B \cdot C}{\|B\| \|C\|}$$

$$\cos \theta_{AC} = \frac{A \cdot C}{\|A\| \|C\|}$$

We know all the quantities on the right hand side here from the problem statement, so finding the inverse cosine of these on a calculator we get the below: I've computed them in degrees as that's a more intuitive unit for some people:

$$\theta_{AB} = 25.4235^\circ$$

$$\theta_{BC} = 89.9986^\circ$$

$$\theta_{AC} = 91.7948^\circ$$

When are two points near each other in space, if all we know is the *angle* between them? Its easiest to think about real life - if you point at two locations in the distance, are those two places close to each other if your arms are making a small angle, or a big angle? A small angle! So, the images that are closest to one another in the space of images are those with a small angle between them as viewed from the origin. The vectors  $A$  and  $B$  are only a 25 degree angle apart, whereas the other two pairs are almost orthogonal to one another. Thus,  $A$  and  $B$  must be two images of the same thing, and  $C$  is an image of something very different!

## Planes and Perpendicularity

Given the planes  $3x+2y-z = 4$  and  $x-z = 16$ , we can easily find a normal vector to each of them by the vector of coefficients: for the first plane, this is the vector  $n_1 = \langle 3, 2, -1 \rangle$  and for the second plane, this is  $n_2 = \langle 1, 0, -1 \rangle$ .

This is of course not what we want in the end at all! We want a vector that lies in both planes. But here's the trick: if you can find a vector  $v$  which is *perpendicular* to  $n_1$ , then its *perpendicular to the vector that's perpendicular to everything in Plane 1*: it must be in plane 1! (This is like the vector calculus version of "the enemy of my enemy is my friend: the perpendicular of my perpendicular is my parallel").

So - any perpendicular to  $n_1$  is parallel to plane 1, and any perpendicular to  $n_2$  is parallel to plane 2. So, if we could find a vector perpendicular to *both of these*, we would have a vector *parallel to both planes*! And we have a tool to do exactly this: the cross product

$$n_1 \times n_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 2 & -1 \\ 1 & 0 & -1 \end{vmatrix}$$

$$\begin{aligned}
&= \hat{i} \begin{vmatrix} 2 & -1 \\ 0 & -1 \end{vmatrix} - \hat{j} \begin{vmatrix} 3 & -1 \\ 1 & -1 \end{vmatrix} + \hat{k} \begin{vmatrix} 3 & 2 \\ 1 & 0 \end{vmatrix} \\
&= -2\hat{i} + 2\hat{j} - 2\hat{k} = \langle -2, 2, -2 \rangle
\end{aligned}$$

This is a vector that's parallel to both planes.

## Shapes

Every point on the plane  $y = 3$  has its  $y$  coordinate...equal to 3! So, to intersect this plane with our sphere we can plug  $y = 3$  into the equation for the sphere, to get an equation taking place just in the  $xz$  plane:  $x^2 + 3^2 + z^2 - 2x - 4(3) + 1 = 0$ . Simplifying this, we see the circle we are interested in is

$$x^2 - 2x + z^2 = 2$$

To figure out what the radius of this circle is, we need to write it in *standard form*: something like  $(x - h)^2 + (z - k)^2 = r^2$ : then we could just read off the radius  $r$ . Thus - we'll need to *complete the square*.

Looking at just the  $x$ 's, we have  $x^2 - 2x$ . We can see that if we had  $x^2 - 2x + 1$  it would factor as  $(x - 1)^2$ , so we can do a little trick of *adding zero in a clever way*:

$$\begin{aligned}
x^2 + 2x &= x^2 - 2x + (1 - 1) \\
&= (x^2 - 2x + 1) - 1 \\
&= (x - 1)^2 - 1
\end{aligned}$$

Substituting this back into the main equation, we see

$$x^2 - 2x + z^2 = (x - 1)^2 - 1 + z^2 = 2$$

Putting all the constants on one side we have managed to reach standard form

$$(x - 1)^2 + z^2 = 3$$

so the radius is  $r = \sqrt{3}$ .



# Assignment 3

This assignment concerns the topics covered in *Parameterization*

## 1. Parametric Curves in the Sciences

GPS technology is frequently used by biologists to track organisms and understand their migration habits. This has particularly been helpful for learning about animals that range over large or uninhabited areas - a fun example being the tracking of great white sharks in the Pacific ocean! Here's a live tracker of some sharks off the coast if you want to see where they are right now. <https://www.ocearch.org/tracker/> Below is the tracking data from a particular shark's locations in the winter of 2014:



Figure 17.11.: GPS tracking data for a Great White off the California coast.

**Exercise 17.7. Part 1:** Let  $\text{shark}(t)$  be the parametric curve giving the location of the tracked shark. What dimension is the output of this curve? (Is it a curve in 2d, 3d, 4d space?). What are some reasonable coordinates you could write this curve in? (Like, what physical meaning could you attach to the different entries of the parametric curve?)

### Assignment 3

**Part 2:** What are some questions that scientists might ask about a great white shark's life? How could you translate these into questions about the parametric curve  $\text{shark}(t)$  so that they could be answered with GPS data?

*There are plenty of different correct answers to this question!*

## 2. Designing Parametric Curves

**Exercise 17.8.** Create a parametric curve  $f(t) = (x(t), y(t))$  in the plane  $\mathbb{R}^2$  looks like the following (it doesn't have to look *exactly like this*)

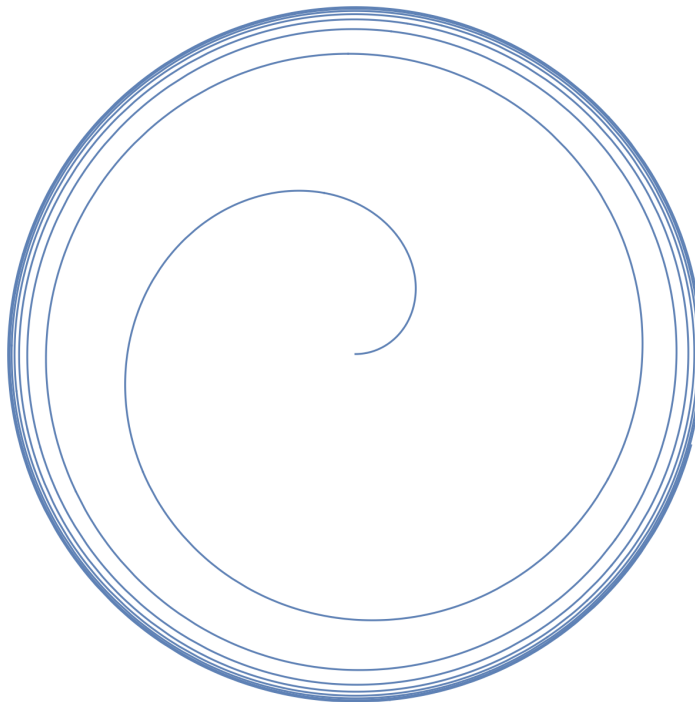


Figure 17.12.: A spiral whose radius asymptotes to 2.

More precisely, find a parametric equation with the following properties:

- As  $t$  increases in  $[0, \infty)$ , the curve  $f(t)$  spirals around the origin at a uniform rate.

### 3. Finding Intersections

- At the beginning  $f(0) = (0, 0)$  and then the spiral approaches a radius of 2 closer and closer, but never quite reaches it as  $t \rightarrow \infty$ .

*Hint: draw a graph of what you want the radius function to look like, then try to build such a function!* There are infinitely many correct answers to this question, but explain your answer and how you came to it.

### 3. Finding Intersections

In class we saw how to parameterize the intersection between two surfaces in three dimensional space. At the beginning, it was helpful to have a picture of the surfaces (like a cylinder and a plane) that we were intersecting to get some intuition, but it turned out we *didn't need a picture at all* if we just tried methodically to build the parametric equation up one step at a time, starting from the equation with less variables and moving to the one with more.

The same tricks work in higher dimensions and are a powerful tool for mathematicians to understand shapes that we can never hope to see. In this problem, you are going to parameterize the intersection between three different *three dimensional spaces inside of four dimensional space!*

**Exercise 17.9.** The three spaces are described in the Cartesian coordinate system  $(x, y, z, w)$  as follows:

- The space  $x^2 + w^2 = 4$
- The space  $w + z^3 = 1$
- The space  $x + y + z = 1$ .

What is a parametric curve  $\vec{r}(t) = (x(t), y(t), z(t), w(t))$  that traces out their intersection in four dimensional space?

*When you write your answer, **do not just give an equation** instead, explain each step of your thought process in full sentences: the goal of this homework is to improve mathematical reasoning and communication skills!*

## Solutions

### Parametric Curves in the Sciences

There are tons of correct answers to this question. Perhaps you modeled the shark's position as a curve into  $\mathbb{R}^2$ , by tracking the latitude and longitude of the shark. Or maybe you wanted to track its location in 3d by recording latitude longitude and depth: then it would be a function into  $\mathbb{R}^3$ . OR - even though time is the input of the curve, maybe you wish to also store the time parameter in our output data: then it would be a curve into 4-dimensional space. Any of these are valid answers, depending on how you want to record things.

Questions you could ask:

- What is the farthest west the shark swam this year? This would become a question about “what is the maximum value of  $y(t)$ ” if  $y$  is the component of the shark's position measuring longitude (east/west direction).
- What is the fastest the shark swam? This would be a question about the speed  $\|\text{shark}'(t)\|$ , and the maximum value of this.
- Did the shark ever stop swimming? This would be a question about the shark's velocity: is  $\text{shark}'(t)$  ever equal to zero?

### Designing Parametric Curves

Because we want the curve to spin around at a constant rate, we know we should start with our usual parameterization of the unit circle  $(\cos t, \sin t)$ . We want to modify this into a spiral, so we will multiply it by a radius function  $r(t)$  to get a curve

$$\vec{c}(t) = (r(t) \cos(t), r(t) \sin(t))$$

The problem gives us some constraints on what  $r(t)$  should be: it needs to be zero at  $t = 0$  and needs to asymptote to 2 as  $t \rightarrow \infty$ . This gives us a good sense of what its graph should look like:

Any function whose graph looks sort of like this will do. There are plenty of possibilities! Here are a couple

- You could remember from calculus that the function  $\arctan(x)$  is zero at  $x = 0$ , has an asymptote at  $\pi/2$ . Thus,  $4/\pi \arctan(x)$  has an asymptote at 2, and would work.



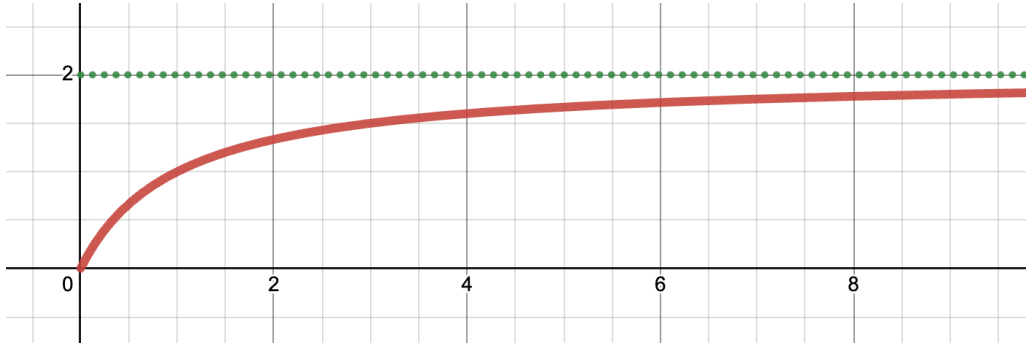


Figure 17.13.: A possible function  $\text{radius}(t)$ .

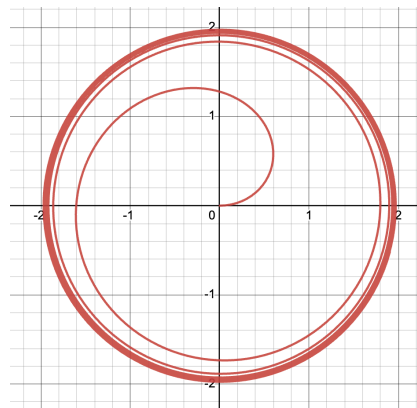


Figure 17.14.: The curve  $\frac{4}{\pi} \arctan(t)(\cos(t), \sin(t))$ .

### Assignment 3

- You could build a function with an asymptote at 2 as a *rational function*, for example  $2x/(x + 1)$ .

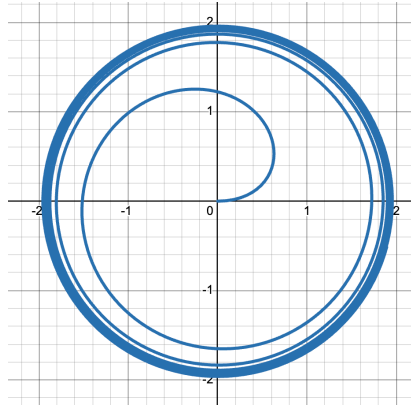


Figure 17.15.: The curve  $\frac{2t}{t+1}(\cos(t), \sin(t))$ .

### Finding Intersections

Here we are given three hypersurfaces in four dimensional space, so we cannot picture exactly what is going on, but we can do the mathematics exactly analogous to what we've done in 2 and 3 dimensions.

The equations we have available are:

$$x^2 + w^2 = 4 \quad w + z^3 = 1 \quad x + y + z = 1$$

The first two each deal with only two variables at a time, so either of these is a good spot to start. Let's start with the first one: we see  $x^2 + w^2 = 4$ , which we recognize as the equation of a circle. Thus we can parameterize these two coordinates in terms of a parameter  $t$  as

$$\begin{pmatrix} x(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} 2 \cos t \\ 2 \sin t \end{pmatrix}$$

Now the second equation involves a  $w$ , so we can substitute what we learned above into it, getting  $2 \cos t + z^3 = 1$ . We can then solve this for  $z$ , to figure out what it must be:

$$z = \sqrt[3]{1 - 2 \sin t}$$

Now we know  $x(t)$ ,  $z(t)$  and  $w(t)$  so all we need to find is  $y(t)$ . But the third equation has a  $y$  in it, so we can solve for  $y$

$$\begin{aligned}y &= 1 - x - z \\ &= 1 - 2 \cos(t) - \sqrt[3]{1 - 2 \sin t}\end{aligned}$$

Putting this all together, we get a parametric curve for all of  $x$ ,  $y$ ,  $z$ ,  $w$ :

$$\vec{c}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} 2 \cos t \\ 1 - 2 \cos(t) - \sqrt[3]{1 - 2 \sin t} \\ \sqrt[3]{1 - 2 \sin t} \\ 2 \sin t \end{pmatrix}$$



# Assignment 4

This assignment concerns the topics covered in *Calculus and Geometry of Curves*

## Problems

### Mission to an Asteroid

This question has a rather involved story. Make sure to understand it fully before starting the problem, and draw yourself lots of pictures!

**Background Story:** One proposed method of preventing a future catastrophe like the impact that killed all non-avian dinosaurs 66 million years ago is *asteroid redirection*, where a spacecraft is launched at the dangerous asteroid and impacts it at a high velocity, slightly deflecting the asteroid's trajectory and causing it to miss Earth. Such a redirect was attempted for the first time last year in 2022, when NASA's DART mission impacted the asteroid Dimorphos, a 500ft rock in a relatively near-earth (but non-dangerous!) orbit. Here's the rather overly dramatic "trailer video" from NASA about the mission:

<https://youtu.be/nHBuVbGkmBY?si=rzDTpWd7X1wbaRoB>

The important science goals of the mission were to:

- (1) see how much the asteroid was deflected after the impact, and
- (2) understand better the physics of high-speed collisions, to be able to model re-direct possibilities more accurately in the future. Objective (1) was able to be accomplished by earth-based telescopes carefully plotting its trajectory in the night sky, but objective (2) required data from near the asteroid itself: ideally actual photographs of the impact moment. Unfortunately the DART spacecraft was unable to send such images back itself, as it impacted the asteroid at a speed of **4.1 miles per second** and was instantly vaporised into plasma.

#### Assignment 4

Here's an actual video shot by the spacecraft on approach. Obviously, the video feed cuts out when the space ship hits the asteroid!

[https://youtu.be/N-OvnVdZP\\_8?si=0ZF1VRallCqNegc5](https://youtu.be/N-OvnVdZP_8?si=0ZF1VRallCqNegc5)

To get the crucial data back to earth, a small pair of italian-built cubesats were carried with DART, and released just days before the impact. These cube sats traveled on a slightly modified trajectory that *just missed* the asteroid, and so were able to take real-time photographs of the impact. One of these photographs is included below. You can clearly see the actual impact (bright spot with debris flying out into space) as well as the asteroid's moon glowing bright in the glow of the molten material created. This image has been extremely crucial in understanding the actual physics of the impact.



Figure 17.16.: The DART spacecraft impacting the asteroid, as viewed from the accompanying cube-sat

**The Problem Statement:** In this problem, you are to imagine that you are a member of NASA's DART team, and are tasked with the planning of these very important shots. Fix a system of coordinates  $(x, y, z)$  (where  $x, y$  and  $z$  are measured in hundreds of miles) such that the asteroid is at the origin for all time. You propose a flight trajectory for the cubesat which departs (at  $t = 0$ ) three days before the impact, which takes the cubesat on the following parametric curve.

$$c(t) = \left( t - 2, \sqrt{t} - \frac{t}{2}, 0 \right)$$

The image below is a schematic of the flight path of the cubesat (red) and the impactor (orange).

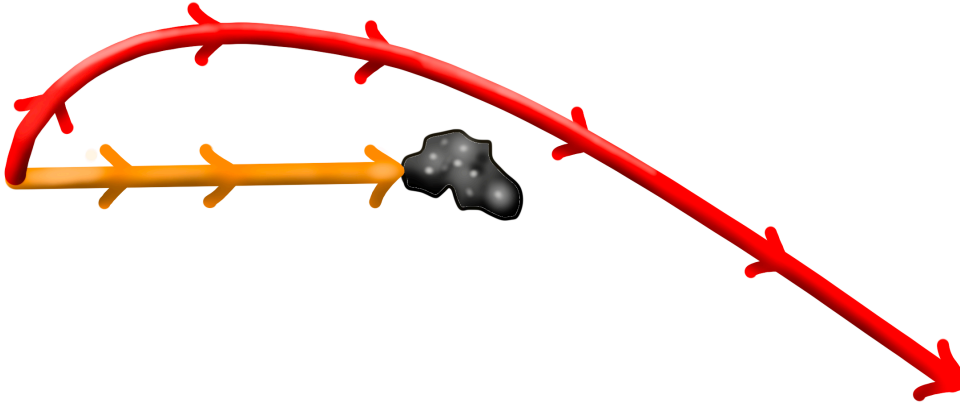


Figure 17.17.: The trajectory of the DART impactor (yellow), and the flightpath of the cubesat (red) that is going to take a photograph.

The cubesat is oriented so that it remains *tangent to its flightpath at all times*.

#### Exercise 17.10.

- At the time of the impact ( $t=3$ ), how far is the cubesat from the asteroid?
- If the cubesat wants to take a picture of the impact, what angle must it rotate its camera by (where angle zero means the camera is pointed straight ahead, tangent to the flight path).

#### Ideas to compute this:

- Can you find the position of the cubesat at the time the asteroid is impacted by the main craft?
- Can you find the direction vector from the cube sat to the asteroid at this time?
- Can you find the direction the cubesat is pointed at this time?
- Can you find the angle between these?

#### Logo Design, Part I

Both this question and the next concerned with using the calculus of parametric curves to help with graphic design. Assume your graphic design firm is hired to produce a new logo for a company, and you decide on the following form:

Assignment 4

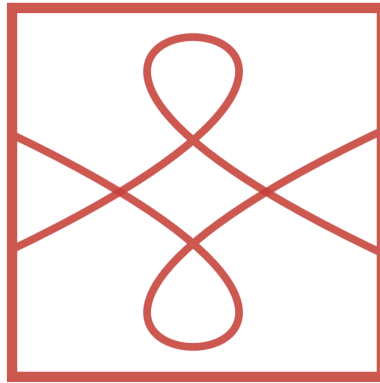


Figure 17.18.: Your proposal for a new fancy logo.

This is made out of a square (the bounding box) and two curves, both of which are copies of the same parametric curve given below

$$\vec{c}(t) = (t^3 - t, t^2)$$

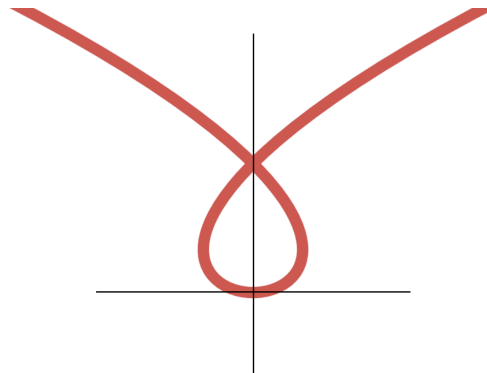


Figure 17.19.: The parametric curve  $\vec{c}(t)$  defining your logo's main pieces.

**Exercise 17.11.** The curve defining your logo makes a sort of teardrop shaped loop in the middle. What is the inside angle of this teardrop?

**Ideas to compute this:**

- Can you find the location where the curve intersects itself?
- Can you find the two times when the curve is at that location?



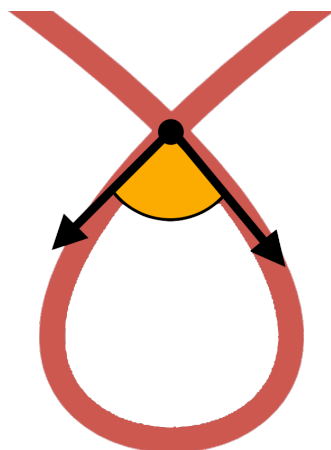


Figure 17.20.: The angle on top of the teardrop.

- Can you find the tangent vectors to the curve there, and make sure they are pointing in the right direction for the angle you want (possibly, negate one of them: think about the direction the curve is traveling in)!
- Can you use the dot product as a tool to get your answer?

## Logo Design, Part II

A lot of thought goes into the design of a logo, and geometric designs are often used to build simplicity and elegance into the final product. Below are some examples of logos, with the underlying geometry on display:

Sometimes the graphic design team goes a little too far into geometry, metaphor, and meaning behind a logo. If you'd like a good laugh check out the [Pepsi Logo Redesign](#)

Like Apple's logo, at every point except the hard corner formed by the intersections (whose angle we computed in the last problem) we can approximate our logo by circles! Drawing a few of these gives the following geometric design (which we may submit to the company that hired us as part of the packet describing the geometry and meaning behind the logo).

Our goal here is to figure out how to actually *compute* these circles. We will focus on only a couple of them, drawn on the curve  $\vec{c}(t)$  itself below.

### Exercise 17.12.

*Assignment 4*

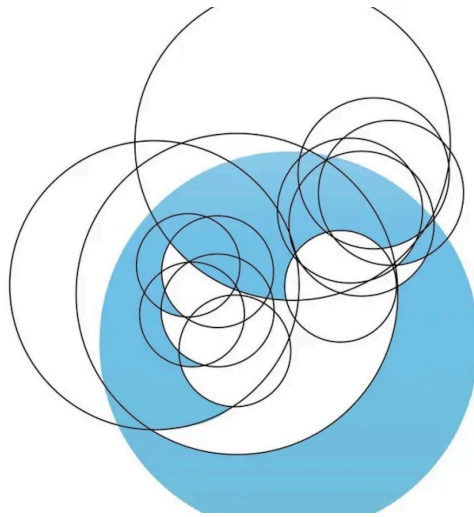


Figure 17.21.: Twitter's old logo was made entirely of circles. Now its made entirely of the letter X.

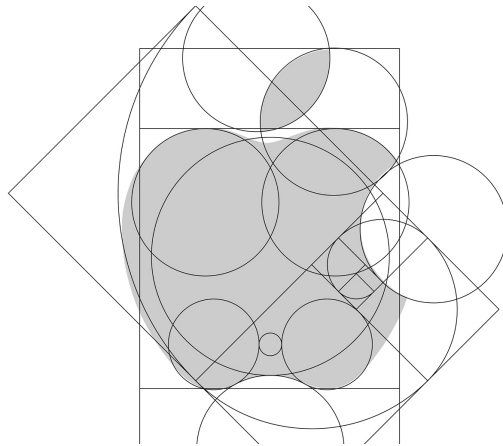


Figure 17.22.: Apples' logo also incorporates many circles, some of which are exact pieces of the logo, and others of which are the "best approximating circles" at a point - such as near the top of the fruit. You can also see the logarithmic spiral ( $e^t \cos t, e^t \sin t$ ) in the design!

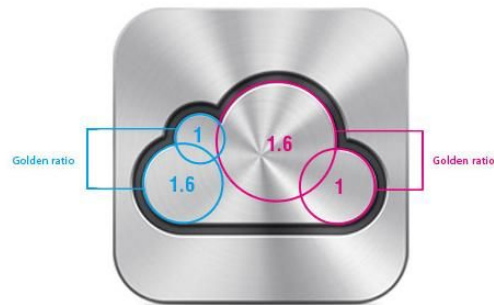


Figure 17.23.: The logo for iCloud is also made of circles, where the ratio of radii is the golden ratio!

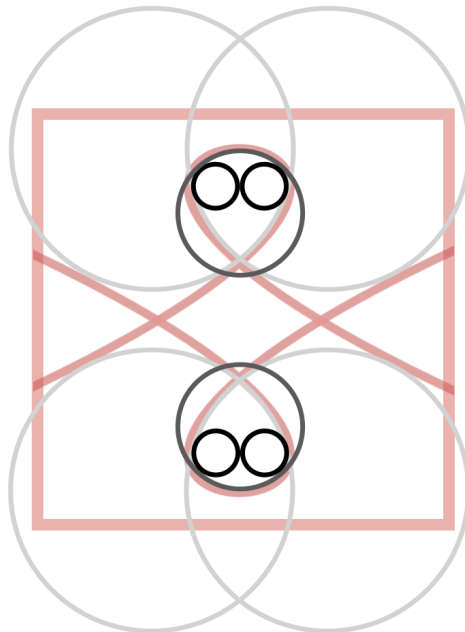


Figure 17.24.: The best approximating circles to our logo at various points, much as in Apple's logo design.

## Assignment 4

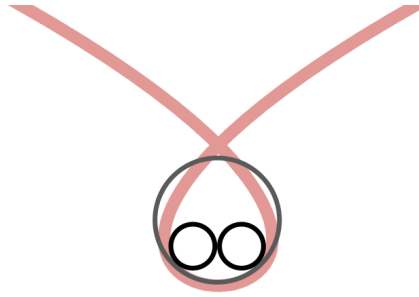


Figure 17.25.: The best approximating circle at the very bottom of the teardrop, and the smallest best approximating circles, which occur at two points along the inside of the teardrop.

- The grey circle below is the circle which best approximates our logo at the very bottom of the teardrop shape. What is its radius?
- The two black circles are the two *smallest circles* that fit inside the teardrop shape. What's an approximate value for their radii?

### Ideas to compute this:

- Can you compute the curvature  $\kappa(t)$  of the logo's curve as a function of  $t$ ?
- For the bottom of the teardrop: what is the value of  $t$  for which  $\vec{c}(t)$  is exactly at the bottom (you should be able to find this by hand, looking at the equation for  $\vec{c}(t)$ ). What is the curvature there?
- What is the relationship between curvature and the size of the best approximating circle?
- Can you put your function  $\kappa(t)$  into desmos, and use the graph to figure out what the radius of the smallest circles must be?

## Solutions

### Mission to an Asteroid

The cubesat is following the trajectory  $\vec{c}(t) = \left(t - 2, \sqrt{t} - \frac{t}{2}, 0\right)$ , so at the time of impact, ( $t = 3$ ) the cubesat is at the position

$$\vec{c}(3) = \left(3 - 2, \sqrt{3} - \frac{3}{2}, 0\right) = \left(1, \sqrt{3} - \frac{3}{2}, 0\right)$$

The *distance from the origin* is just the magnitude of this vector:

$$\begin{aligned} \text{dist} &= \sqrt{(1)^2 + \left(\sqrt{3} - \frac{3}{2}\right)^2 + 0^2} \\ &= \sqrt{1 + 3 - 3\sqrt{3} + \frac{9}{4}} \\ &= \sqrt{\frac{25}{4} - 3\sqrt{3}} \\ &\approx 1.0265707 \end{aligned}$$

To find the angle we need to rotate the camera, we need to first know two vectors:

- The vector **tangent** describing the direction the cubesat is pointed in
- The vector **toAsteroid** describing the direction from the cubesat to the asteroid.

We begin with the second of these. The asteroid is at the position  $O = (0, 0)$  for all time. Thus, the direction vector from the cubesat to the origin is

$$\begin{aligned} \text{end} - \text{start} &= O - \vec{c}(3) \\ &= (0, 0, 0) - \left(1, \sqrt{3} - \frac{3}{2}, 0\right) \\ &= \left(-1, \frac{3}{2} - \sqrt{3}, 0\right) \end{aligned}$$

To find the direction the cubesat is facing, we need to find the *tangent vector* to its trajectory  $\vec{c}(t)$  at  $t = 3$ .

$$\begin{aligned} \vec{c}'(t) &= \left(1, \frac{1}{2\sqrt{t}} + \frac{1}{2}, 0\right) \\ \vec{c}'(3) &= \left(1, \frac{1}{2\sqrt{3}} + \frac{1}{2}, 0\right) \end{aligned}$$

Now, we just need to find the angle between these two vectors. To do this, we can use the identity

$$\cos \theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\| \|\mathbf{v}\|}$$

Since the vector pointing at the asteroid is the *negative* of the position vector of the cubesat, its magnitude is the same as the distance to the asteroid, that we already found.

## Assignment 4

$$\|\text{toAsteroid}\| \approx 1.0265707$$

The magnitude of the tangent vector is

$$\|\text{tangent}\| = \sqrt{1 + \left(\frac{1}{2\sqrt{3}} + \frac{1}{2}\right)^2} \approx 1.273580962$$

Finally we need their dot product

$$\begin{aligned}\text{toAsteroid} \cdot \text{tangent} &= \left(-1, \frac{3}{2} - \sqrt{3}, 0\right) \cdot \left(1, \frac{1}{2\sqrt{3}} + \frac{1}{2}\right) \\ &= (-1)(1) + \left(\frac{3}{2} - \sqrt{3}\right)\left(\frac{1}{2\sqrt{3}} + \frac{1}{2}\right) \\ &\approx -1.1830\end{aligned}$$

Putting these together we can compute the cosine of the angle:

$$\cos \theta \approx \frac{-1.1830}{(1.27358)(1.0265)} \approx -0.9053$$

This means that  $\theta \approx \arccos(-0.9053)$  which is 2.7029 radians, or **154.868 degrees**.

## Logo Design I

The looped part of the new logo design is described by the parametric function  $\vec{c}(t) = (t^3 - t, t^2)$ . First, we try to understand what direction this curve is being traced out by the parameterization. When  $t = -2$  the  $x$ -coordinate of our function is  $(-2)^3 - (-2) = -8 + 2 = -6$ , whereas when  $t = 2$  its  $(2)^3 - 2 = 6$ . Thus, as  $t$  increases we see the  $x$  coordinate going from negative to positive, so the curve is being traced from left to right.

*Another way to have done this part is just to look at the two equations and see that  $t = \pm 1$  gives the same answers for each!*

To find the spot where the loop crosses itself, we need to find two different  $t$  values that reach that same  $(x, y)$  point on the curve. Because the  $y$  coordinate  $y = t^2$  is an even function it maps  $t = a$  and  $t = -a$  to the same point for any  $a$ . But since  $x(t) = t^3 - t$  is an *odd function* we have  $x(-a) = -x(a)$ . And the only time  $x(a)$  can equal  $-x(a)$  is if they are both zero (that's the only number equal to its own negative)! Thus, we know  $x(a) = 0$  so  $a^3 = a$ , or  $a = \pm 1$ .

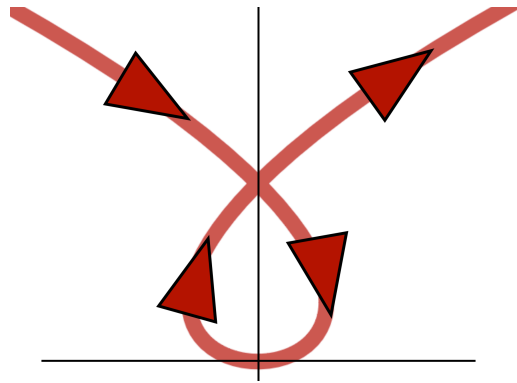


Figure 17.26.: The direction the parametric curve is traced.

These are then our two time values,  $t = -1$  and  $t = 1$ , which both map to the point  $c(1) = (0, 1)$ . To find the angle of the teardrop at this point, we need to find the derivative of  $c(t)$  and both  $t = -1$  and  $t = 1$ .

This is just a calculation

$$\begin{aligned}\vec{c}'(t) &= \langle 3t^2 - 1, 2t \rangle \\ \vec{c}'(-1) &= \langle 2, -2 \rangle & \vec{c}'(1) &= \langle 2, 2 \rangle\end{aligned}$$

These are the *velocity vectors* of the curve at the two points it crosses through  $(0, 1)$ . To help us out, let's draw them quick, remembering the direction the curve is being traced.

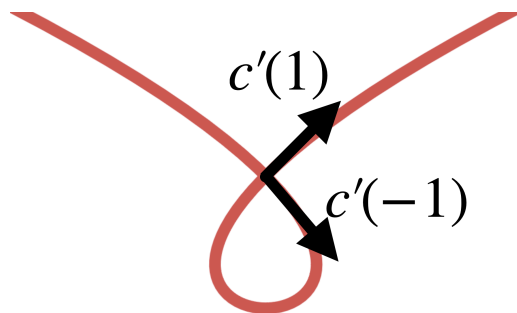


Figure 17.27.: The tangent vectors at  $t = \pm 1$ .

To find the angle of the teardrop, we need the two vectors tangent to the teardrop's angle. From the above, we see that the tangent vector at  $t = -1$  is in the correct direction, but the one at  $t = 1$  is in the wrong direction: we need to negate it! Thus, the two vectors

### Assignment 4

we are interested in are  $\langle 2, -2 \rangle$  and  $\langle -2, -2 \rangle$ . Again using the relationship of angles to the dot product,

$$\begin{aligned}\cos \theta &= \frac{\langle 2, -2 \rangle \cdot \langle -2, -2 \rangle}{\|\langle 2, -2 \rangle\| \|\langle -2, -2 \rangle\|} \\ &= \frac{-2 + 2}{\sqrt{8}\sqrt{8}} = 0\end{aligned}$$

Thus,  $\theta = \pi/2$  or 90 degrees.

## Logo Design II

The curvature of a curve is the reciprocal of the radius of the best approximating circle. Thus, the first step here is to find the curvature of our logo's curve. For this, we can use the formula relating curvature to the curve's first and second derivatives:

$$\kappa(t) = \frac{\|c'(t) \times c''(t)\|}{\|c'(t)\|^3}$$

We have computed the first derivative above, and differentiating again gives both velocity and acceleration.

$$c'(t) = \langle 3t^2 - 1, 2t \rangle$$

$$c''(t) = \langle 6t, 2 \rangle$$

We find the cross product of the two vectors we add in a zero for the  $z$  component, and use the determinant formula.

$$\begin{aligned}c'(t) \times c''(t) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3t^2 - 1 & 2t & 0 \\ 6t & 2 & 0 \end{vmatrix} \\ &= \hat{k}((3t^2 - 1)(2) - (2t)(6t)) \\ &= -(6t^2 + 2)\hat{k}\end{aligned}$$

Then the magnitude of this,



$$\begin{aligned}\|c' \times c''\| &= \|\langle 0, 0, -6t^2 - 2 \rangle\| \\ &= 6t^2 + 2\end{aligned}$$

and also the magnitude of  $c'$ .

$$\begin{aligned}\|c'(t)\| &= \sqrt{(3t^2 - 1)^2 + (2t)^2} \\ &= \sqrt{9t^4 - 2t^2 + 1}\end{aligned}$$

Now we have all the components, and can assemble the curvature!

$$\kappa(t) = \frac{6t^2 + 2}{(9t^4 - 2t^2 + 1)^{3/2}}$$

Phew! Now we can finally answer the questions we are interested in. To answer the first, we need to find the curvature at the bottom of the loop. This is directly below where the curve intersects itself, which we learned in the previous part occurs at  $x = 0$ . Thus this point must also be at  $x = 0$ : the only other time value that gives this is  $t = 0$ , so this is where we need the curvature!

$$\kappa(0) = \frac{6(0)^2 + 2}{(9(0)^4 - 2(0)^2 + 1)^{3/2}} = 2$$

Thus, the radius of the best approximating circle is

$$R = \frac{1}{2}$$

For the second part, we are looking for the *smallest* approximating circles inside the teardrop, which means we want to find the *maximum* value of curvature between  $t = -1$  and  $t = 1$ . This function does not look fun to differentiate and set equal to zero, so we will take the problem's suggestion and put it into desmos to look for the max:

Desmos shows that the maximum occurs around  $t = 0.422$  with  $\kappa \approx 3.425$ , so the radius of the smallest circles is

$$R = \frac{1}{\kappa} \approx 0.2919$$

Assignment 4

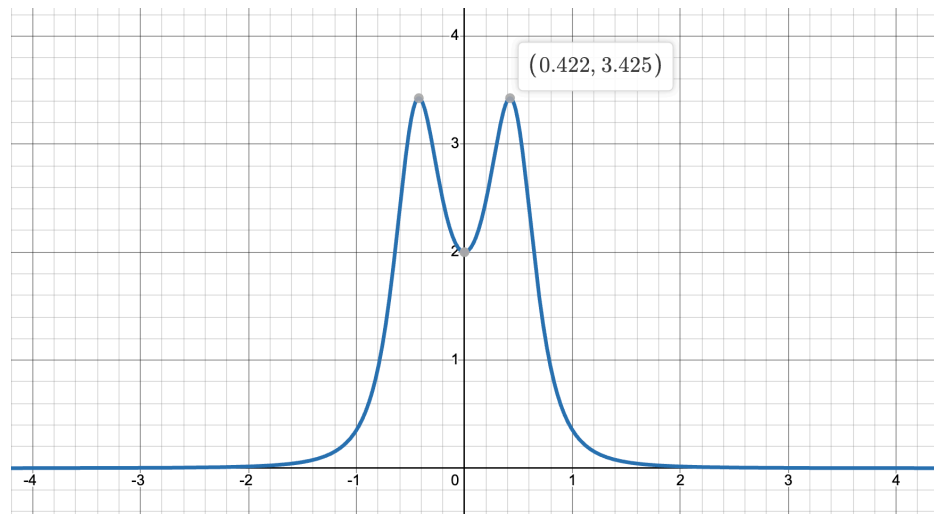


Figure 17.28.: The curvature function  $\kappa(t)$

# Study Guide I

Write a *study guide* for yourself in preparation for Midterm I.

**You can write the guide however helps you most**, but here are some possible suggestions:

- Write down the important definitions for each topic you want to review. Explain them in your own words!
- Either come up with an example problem (or find one on WebAssign) for each topic you want to review, and write out its solution by hand! You'll be handwriting solutions on the exam, so this is a good way to get yourself in that mindset and away from online work.
- For each topic you want to review, write to yourself how confident you are with it (maybe on like a 1-10 scale, if that's helpful to you): this way when you're studying you can look back and easily remember which things you were struggling with.
- If there's multiple ways to do something, make a note of that for yourself!

Below I've made a short list of the topics we've covered so far which could appear on the exam. (This is essentially a table of contents for the first two sections of our notes, but will be helpful to those who are primarily reading the textbook instead). I will ensure test problems only cover the below topics.

## Space

### Dimensions

- What is the dimension of a space? Can you calculate the dimension of various example spaces (surface of the earth, a space of images, configuration space of a robot, etc).
- Be comfortable using cartesian coordinates and  $n$ -tuples to represent points in space. Understand polar and cylindrical coordinates as alternative methods to represent points using  $n$ -tuples.
- Be able to use the distance function in  $\mathbb{R}^n$ .

## Vectors

- Know the difference between a point (location) and a vector (direction + magnitude). Be able to give examples of quantities that are points versus vectors.
- Be able to use vector addition and scalar multiplication. Understand how they work geometrically as well as in terms of cartesian coordinates.
- Be able to compute the magnitude of a vector, find the unit vector in a given direction, and combining these - find a vector of a given length in the direction of another vector.
- Be familiar with the two notations for vectors using  $n$ -tuples  $\langle x, y, z \rangle$  as well as  $ijk$  notation  $x\hat{i} + y\hat{j} + z\hat{k}$ .

## Operations

### Dot Product

- Know the formula for the dot product, and be able to use it. Know how the dot product interacts with vector addition and scalar multiplication.
- Understand the relationship between the dot product and angles. Be able to find the angle between two vectors (you will not have a calculator, so it is fine if your answer looks like  $\arccos(3/4)$  or something on the test.) Know the definition of orthogonality, and how to test if two vectors are orthogonal with the dot product.
- Be able to compute the scalar and vector projections of one vector onto another.

### Cross Product

- Know how to compute the cross product using determinants. Be able to compute the cross product of the standard basis vectors  $\hat{i}, \hat{j}, \hat{k}$  using the “circle” diagram.
- Understand the geometric meaning of the cross product, the right hand rule, and the connection between the length of the cross product and the area of a parallelogram.

**I will not test you on:** The vector triple product  $a \cdot (b \times c)$  - even though you do already know the dot product and the cross product!

## Shapes

- Know both the implicit equations and parametric equations for a line. Be able to find the equation of a line given two points, or a point on the line and a direction vector. Be able to find the normal vector to an implicit line.
- Know both the implicit equations and parametric equations for a plane.
- Be able to find the normal vector to an implicit plane and to a parametric plane.
- Be able to find the equation of a plane (whichever form is easier for you) given some information (for example: a point and a normal vector, or a point and two vectors, or three points)
- Know the implicit and parametric equations for a circle of radius  $R$  centered at  $(h, k)$ .
- Know the implicit equation for spheres in  $\mathbb{R}^3$ , and be able to find their center and radius (perhaps by completing the square)
- Understand the relationship between the equation for a circle in 2D and a cylinder in 3D.

**I will not test you on:** The parametric equation for a sphere, or the other shapes we briefly talked about (hyperboloids, ellipsoids, etc.)

## Curves

### Parameterizations

- Know the definition of parametric curves in both 2 and 3 dimensions. Be able to parameterize graphs of functions.
- Understand that there may be multiple different parameterizations for the same curve, and be able to tell when this happens (if the coordinates of each parameterization satisfy the same relationship: like  $(t, t)$  and  $(t^3, t^3)$  both satisfy  $y = x$ ).
- Be able to parameterize the intersection of surfaces in space, by using the equations of the surfaces one at a time to solve for  $x, y, z$  in terms of a parameter  $t$ . (Example: intersection of a cylinder and a plane, from class)
- Understand how operations like scalar multiplication affect the shape of a parametric curve. Be able to turn a circle into a spiral or a helix, etc by changing the parameterization slightly.

### Calculus

- Know how to take limits of a parametric curve (via the limit of each component)

## Study Guide I

- Know why you can take the derivative of a parametric curve component by component (applying the limit definition of the derivative, and then breaking it up into individual limits using the first point.)
- Understand the geometric meaning of the derivative (a tangent vector to the curve) as well as its physical meaning (the velocity the particle tracing out the curve is moving), and of its magnitude (the speed).
- Know how to differentiate vector functions with different products (we have three products now, scalar multiplication, the dot product, and the cross product - so we have three product rules)!
- Be able to integrate a vector function, and understand its meaning: recovering velocity from acceleration, or displacement from velocity.

## Geometry

- Be able to write down the arc length integral for a curve. Be able to compute the actual arclength when this integral is simple.
- Determine whether or not a curve is parameterized by arclength by checking if its derivative is unit length.
- Find the unit tangent vector to a curve.
- Know the definition of the normal vector to a curve, and be able to find the normal vector in simple cases.
- Know the formula for the curvature of a parametric curve  $\vec{r}$  in terms of its first and second derivatives. Understand how we were able to simplify this when the curve was a graph  $r(t) = (t, f(t), 0)$ .
- Understand the relationship between curvature and the radius of the best approximating circle.

## Differential Equations

- Know what a vector valued differential equation is, and how to interpret it as a system of differential equations.
- Understand how to derive that objects fall on parabolas under a constant downward gravitational acceleration. Be able to solve simple differential equations like this (where the individual functions do not depend on one another.)
- Be able to take an *almost solution* and adjust it to make it an actual solution by changing its speed (like we did for circular orbits in Newtonian gravity)

Because this topic was our most recent topic (and thus there is no webAssign questions on it in our previous homeworks), I wanted to give a couple example questions below, that could be reasonable to ask.

- If  $c = (x(t), y(t))$ , find a solution to  $c' = \langle 2t, e^t - 1 \rangle$
- Show that  $c(t) = (\cos(t), \sin(t))$  solves the differential equation  $c'(t) \cdot c'(t) = 1$ .
- The curve  $(x(t), y(t)) = (t, t^2)$  is almost a solution to the equation  $xy' = y$ . Can you adjust one of the component functions by a constant to make it a solution?





# Assignment 5

## Problems

### Reading Contour Plots

You are hired as a consultant for an aerospace company that is working to design the next generation of rocket boosters. The main goal of your team is to improve the *efficiency* of the engine (the total impulse delivered per kilogram of fuel). You begin by focusing on the two most important variables: the rocket engine nozzle size  $n_s$ , and the temperature  $T$  of the burning fuel. You write a computer program to simulate thousands of different combinations of temperature and size, and compute the efficiency  $E_{ff}(T, n_s)$  of each configuration. After a lot of computation, your program outputs the following figure:

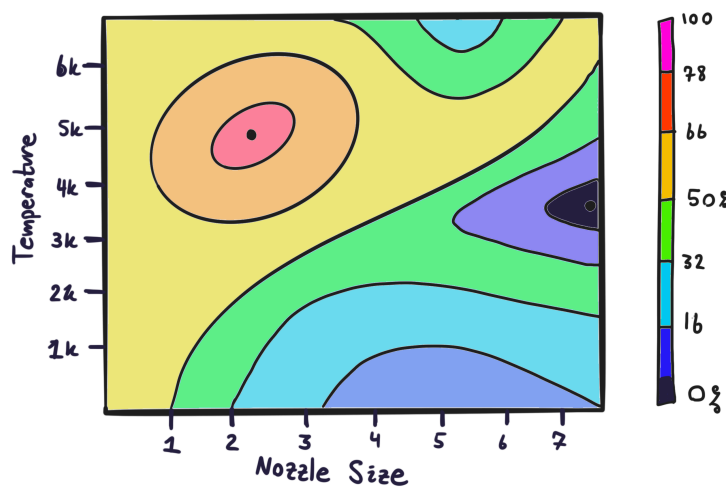


Figure 17.29.: A multivariate function of efficiency for rocket engine design.

How do you summarize your findings to your boss, given this data? An example summary might sound something like this (though this is not the answer for this exact plot, of course!)

## Assignment 5

For the most part, rocket engines perform better when the nozzle size is large and the temperature is small, though if the temperature gets too low it starts performing worse again. Performance is particularly bad for temperature of 300 and small nozzle sizes (around 1). The most efficient configuration is nozzle=5 and temp=3.2k.

If the current rocket engines in production have a nozzle size of 1 and a fuel temperature of 6k, what concrete suggestion could you make to your team on how to change the design slightly to improve efficiency? What would you say to a team currently building a rocket with nozzle size 7 and temperature 3k?

**Example questions you may want to consider:** *Should you increase both temp and nozzle size? Decrease one of them and leave the other constant? Decrease both? Should they make big or small changes to their current design? Etc...*

## Partial Derivatives

The data of multivariate functions can be presented to us in multiple ways. Sometimes, we are given a contour plot - for example, as the output of a simulation like in the problem above. But other times we are just given an *equation*, and have to work with that.

In this problem we will continue our roll as an aerospace consultant engineer, working on the same rocket design as above. After successfully helping with the engine design, you are asked to work closer with the team that is building the rocket's body so that you can safely integrate the engine. This team is run by a collection of theorists who have been able to work out a model for the rocket's aerodynamic efficiency in terms of the angle  $\alpha$  and height  $h$  of its fins, the radius  $r$  of the main tube, and the thickness  $t$  of the metal support beams. Their model is

$$E = re^{-r/2} \sin(h\alpha/t)$$

Because of your success in optimizing the engine design, this team asks you for your help in optimizing the rocket body. In particular, they are currently considering building a rocket body with radius 1.5, beam thickness of 0.3, fin angle of  $\pi/4$  and fin height of 1.

They want to know **should they increase or decrease the beam thickness, to improve efficiency?** Because the rocket body design depends on so many parameters, we can't draw its graph (which lives in 5 dimensional space) or its contour plot (which lives in four dimensions!). So, we need to *just do some calculus*.

## Graphs of Multivariate Functions

You are working with a real-estate developer to plan new environmentally-friendly infrastructure for a community in the rolling hills of Marin county. One particular hill you are developing is well approximated by the graph of the quadratic function

$$z = 5 - \frac{x^2}{4} - \frac{y^2}{8}$$

For  $(x, y) \in [-2, 2] \times [-2, 2]$ .

Your job is to find the optimal placement along the hill for the collection of solar panels that will provide electricity for the development. Solar panels work best when the sun light shines directly perpendicularly onto their surface, so you are looking for the location on the hill where the sun hits it most perpendicularly.

Consulting a local meteorologist you learn the direction of the average noontime sun at your build site is  $\langle 1, 0, -4 \rangle$ . At which point  $(x, y, z)$  should we build the solar panel farm? To figure this out, follow the outline below.

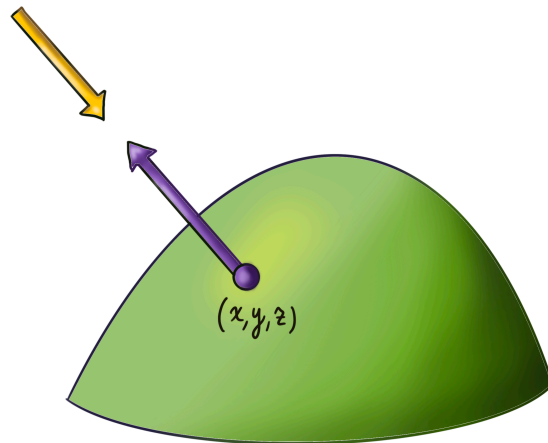


Figure 17.30.: Finding the optimal location on a hill to place solar panels requires understanding the sun's direction.

- At any point  $(x, y)$  on the plane, can you use partial derivatives to find a vector which is perpendicular to the hill at that point?
- Our real goal is to find when the hill is perpendicular to the direction of the sun. For which  $(x, y)$  is the vector you found parallel to this?

## Assignment 5

# Solutions

## Reading Contour Plots

This plot shows that rocket engine efficiency is maximized around a nozzle size of 2, and a temperature of 5k. Small changes from this configuration in any way decrease the efficiency somewhat. The worst performing rocket engines are those with large nozzle size, and temperature around mid-range (3k degrees).

If the team is currently building a rocket with nozzle size 1 and temperature 6k, they should slightly decrease the temperature while increasing nozzle size, in order to best increase efficiency.

## Partial Derivatives

A function is *increasing* if its derivative is positive, and *decreasing* when the derivative is negative. Here we want to know if the efficiency is increasing or decreasing as we change the beam thickness, so we need to take the *partial derivative* in the direction of the variable  $t$ .

Computing this,

$$\partial_t E = \partial_t \left( r e^{r/2} \sin \left( \frac{h\alpha}{t} \right) \right) = r e^{-r/2} \cos \left( \frac{h\alpha}{t} \right) \left( \frac{-h\alpha}{t^2} \right)$$

Next, because we are only interested in what is happening *near* the actual rocket configuration being built, we must plug in the current rocket parameters of  $r = 1.5$ ,  $t = 0.3$  and  $\alpha = \pi/4$  to get

$$\begin{aligned} \partial_t E &= 1.5 e^{-0.75} \cos \left( \frac{(1)(\pi/4)}{0.3} \right) \left( \frac{-(1)(\pi/4)}{0.3^2} \right) \\ &= 0.70854 \cdot (-0.8660) \cdot (-8.7266) \\ &= 5.354603 \end{aligned}$$

This is positive, so we should *increase thickness* if we want to improve the efficiency of our current rocket design.

*In fact this tells us even more quantitative information. Since this value 5.35 ... is the slope this says if we increase thickness by a small amount like 0.1 then the efficiency increases by  $5.35 \cdot 0.1 = 0.535$ .*

## Graphs of Multivariate Functions

In this problem we know the *direction* of the sun's rays, and we want to know when the normal vector to the hill is parallel to this: that is when is the hills normal vector a scalar multiple of  $\langle 1, 0, -4 \rangle$ ?

The first step is to find the normal vector to the hill  $z = 5 - \frac{x^2}{4} - \frac{y^2}{8}$  at the point  $(x, y)$ . In class, we found the normal vector to the graph of a multivariate function to be the *coefficient vector* from its tangent plane:

$$n = \langle f_x, f_y, -1 \rangle$$

This version is *downward pointing* as the  $z$  component is  $-1$ , so looking at our picture, this normal vector is pointed *into the hill*.

To find the actual vector for this problem, we need to calculate these derivatives:

$$f_x = -\frac{x}{2} \quad f_y = -\frac{y}{4}$$

$$n = \left\langle -\frac{x}{2}, -\frac{y}{4}, -1 \right\rangle$$

To figure out when this is a multiple of  $\langle 1, 0, -4 \rangle$  we are just asking when there is a single constant  $c$  such that

$$k \begin{pmatrix} -x/2 \\ -y/4 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -4 \end{pmatrix}$$

This is a system of three equations that we need to solve. The bottom equation tells us that  $k = 4$ . Substituting this into the second equation, we see  $-ky/4 = 0$  implies  $y = 0$ , and into the first equation yields

$$-kx/2 = -4x/2 = 2x = 1 \implies x = -1/2$$

Thus the point where the normal vector is parallel to the sunrays - where we should place our solar panel on the hill has  $(x, y)$  coordinates  $(-1/2, 0)$ .

*Alternatively - if we did not remember this we could derive it using our multivariate tools of partial differentiation and the cross product! Here we could find two vectors that are tangent to the hill using partial derivatives: if you move infinitesimally one unit in the  $x$  direction your  $z = f(x, y)$  component will move by  $\partial_x f$ , and so the vector*

$$v_x = \begin{pmatrix} 1 \\ 0 \\ f_x \end{pmatrix}$$

*is tangent in the  $x$ -direction. Similarly,  $v_y = \langle 0, 1, f_y \rangle$  is tangent in the  $y$  direction, and their cross product is normal:*

$$n = v_x \times v_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \begin{pmatrix} -f_x \\ -f_y \\ 1 \end{pmatrix}$$

*This normal is upward pointing at  $z$  is positive: looking at our picture its pointed out of the hill.*



## Reflection & Correction

The goal of a college course is always *learning* - with homework, exams, and other assignments all just vehicles (and motivation) to help learning occur. Thus, my main goal from this exam is to help you all *learn* where you currently stand in the course, and to set yourself up for success going forward.

- The first part of this process was the *studying for the exam*: where you reviewed all the material on vectors and curves. Here you could earn extra points towards your grade (if you want to use the hold-me-accountable-grading) by **writing a study guide for yourself**.
- The second part of the process was *taking the exam itself*. Here the goal is for both you and me to learn what you know so far in the course, and gauge what needs to be done going forward.

Now, we are at the *post-exam-period* where we try to use the test to learn from our pasts. To really get set up on the right path requires some work, so there will be a *two opportunities to earn points* here as motivation.

- First - you can do an **extra credit exam correction**. You will be able to regain the *full amount of points missed on one problem of your choice*. That is - if you got a 2/10 on one problem you can earn back 8 points if you do a correction on that problem.
- The second, is to **write an exam reflection** as part of the hold-me-accountable grading system. Remember this grading system gives weight to other assignments like this, and in the process lowers the amount that exams count towards your final grade.

### Extra Credit: Exam Correction

**For 100% of the missed points back any problem of your choice:** Write a *study guide* for the problem, addressed to your past self, (or your future self, when you review

## Reflection & Correction

for the final). **MAKE SURE TO INCLUDE BOTH THE PROBLEM STATEMENT, AND THE ORIGINAL SCORE YOU RECEIVED**

You are not simply fixing the mistakes in your old solution, but writing a rather involved document *teaching the mathematics* necessary to succeed at questions like this. Your submission should be *neatly hand written* or *typed* and in full paragraphs with complete sentences. It should not be a rough draft, or an outline (bullet-point list of thoughts, etc). **Complete submissions will likely be several pages in length.** Below is an outline to help you structure such a lesson.

**Analysis of your original solution** What is the complete statement of the problem you are writing a study guide for? When you were working on the exam, what did you think about or try? If you were stuck at the beginning, or did not write much, what made it difficult to make progress?

**Teach the necessary techniques** Before you begin writing up your correction, learn how to solve the problem in its entirety. Look at your solution, and identify the main tools you needed: do you need partial derivatives? Parametric curves? Dot and cross products of vectors? Were some of your struggles caused by material prior to our course? (Differentiation rules, or algebraic manipulation rules?). Were your difficulties conceptual (had trouble picturing the difference between a vector and scalar function, unsure how to tell if something is a circle or cylinder) or computational (forgot the formula for cross product, etc)?

**For each mathematical technique that is integral to the solution of this problem** write a subsection reviewing this technique. Your section should include

- The technique itself (the formula / identity, or set of rules used)
- A discussion of what this technique is *for*: when should your reader (future you) think to use it in a problem?
- An example (or two, or three) of using this technique correctly, in simple problems.
- A discussion of potential pitfalls: what are situations where using this mistake can easily lead to a mistake? (For a calc 2 example; forgetting to convert  $dx$  in  $u$ -sub, or some of the more involved conversions back to  $x$  from  $\theta$  for trigonometric substitution)

**A Full Solution of the Problem** Now that you have taught your reader all of the mathematics necessary to do this problem, write up a *complete, annotated solution*. Start by repeating the problem statement, and then talk your reader through (in sentences) what you should be thinking about at each step. *I know that I have posted the exam solutions! So I am not asking you to just write symbol for symbol what I wrote. I am asking you to really explain what is going on in the solution.* Write this as though you



are truly trying to help your future self feel confident about this problem when they are reviewing this material down the road.

**A similar problem** Now that you have become an ‘expert’ at this one problem, **make up a new example question that is similar** (as in, it uses the same techniques to solve). To make sure it is indeed similar - you should solve it after proposing it! But you do not need to include the solution in your writeup. Instead, this problem will be waiting for you to try again next time you study.

## **Hold-Me-Accountable: Exam Reflection**

Reflect on the the first part of the semester, focusing on your studying techniques, your exam performance, and suggestions to your future self. Your submission should be *neatly hand written* or *typed* and in full paragraphs with complete sentences. It should not be a rough draft, or an outline (bullet-point list of thoughts, etc). There are no wrong answers, but only submissions showing real work at introspection will receive credit: remember, this is an (optional) opportunity for you to think about what works best *for you*

**How did the exam go?** While it is still fresh on your mind, think about the exam itself. How did you do compared to how you expected to do (after studying, but before the exam itself)? After getting feedback, how did the exam go relative to how you felt after taking it? Did you do better or worse than you felt you had (after leaving the exam, but before getting feedback).

If you did well - what were the big contributions to that success for you? If you hoped to have done better, what were some factors that may have affected your performance (these include comfort level with the material, but also things like not sleeping enough the day before, or time pressure etc).

**What were your study strategies?** How did you go about preparing for this exam? Did you redo homework problems? Re-read lecture notes? Go back through the chapters in the book? Did you study for the exam with friends, or alone? How did you use the practice exam? Of the things you did do, what felt like it had the most payoff? Did any of your studying feel unproductive for you (as in, you put a lot of time into a particular concept or strategy, but in the end still struggled with that)?

**How did you learn outside of class?** Do you do practice problems beyond the homework? And if so, how do you decide what topics you need more problems for?

If not, how can you begin to use the homework to your advantage, as a means of helping you identify what the most difficult points in a given week are? When reviewing a new concept, do you spend time reading the book chapters that accompany lecture, or

### *Reflection & Correction*

watching internet videos on these topics (or both, or neither)? Does the amount of time you spend on this class match its relative difficulty? (ie are you spending alot of time because its one of your hardest classes, or little time because its easy? This would mean it does match. If you are spending not much time but its one of your harder classes, this would not match.) This is a four credit course - the university expects students to spend *eight hours outside of classtime* studying for the course.

Reflect on what you can do each week going forward to ensure that you are using the homework to your advantage: are you not finishing problems and so wish you could find a bigger study group to work with? Are you relying too much on friends and coming away from problems with an incomplete understanding (even though you get them correct)?

**What are some recommendations for your future self?** If you could help your past self set up a strategy to leverage your own strengths (and efficiently identify your points of struggle) for the first third of this class, what would you do? What parts of your current strategy will you keep, and what would you have changed? Be realistic (ie don't just say "I would study X hours more" if adding X hours to your current plan leaves you with an unsustainable work balance across your classes), and take what you have learned about yourself in the above questions to build a reasonable plan.

# Assignment 6

## Problems

### Heatshields on Rockets

You are working with several research teams at NASA who are trying to understand and model the heat flow along the nose cone of a rocket during re-entry. Below is a picture from a real simulation of this just for context (though you do not need this picture to solve the problem)

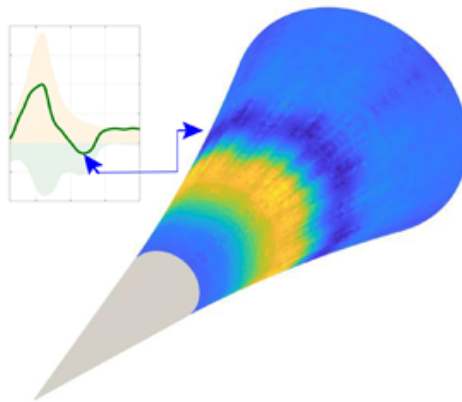


Figure 17.31.: Waves of hot and cold on a nose-cone for a rocket traveling at hypersonic speeds.

You and your team use the heat equation as a model of this: if  $x$  is the distance along the nose cone (starting with  $x = 0$  at the tip) and  $t$  is the time, then the temperature  $H(x, t)$  of the metal at point  $x$  and time  $t$  satisfies the equation

$$\frac{\partial H}{\partial t} = \lambda^2 \frac{\partial^2 H}{\partial x^2}$$

## Assignment 6

Where  $\lambda$  is a positive constant that depends on the metal you choose to make the nose of the rocket out of (larger values of  $\lambda$  means that it conducts heat *faster*, smaller values mean it conducts heat *slower*).

Your team of atmospheric scientists tell you that during re-entry *waves of plasma* will be crashing over the nose cone, heating it up in a complicated pattern. Once this portion of the flight is over, the temperature distribution along the nose cone is expected to look like

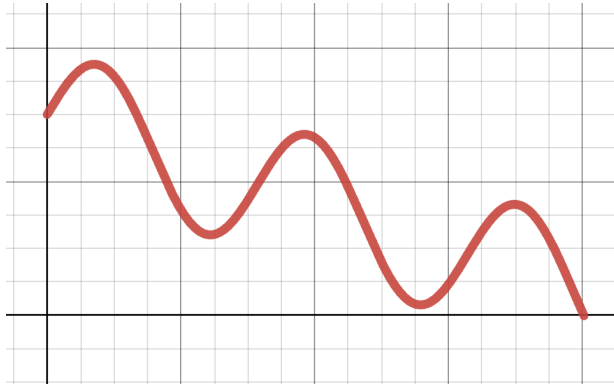


Figure 17.32.: Temperature distribution (as a function of  $x$  along the nose cone) at time  $t = 0$ .

- Explain the heat equation in words: how does it relate the change in heat over time to the distribution of heat in the  $x$  direction? Using the heat equation, draw a picture of what the heat distribution on the nosecone will after a short time. Explain your graph: why did you draw it like this?

Your team of theorists find an exact solution to the heat equation describing these plasma waves: it looks like this

$$H(x, t) = e^{-\lambda^2 t} \sin(\lambda x)$$

(You don't have to turn this in, but it's a good idea to verify this is a solution to the heat equation, by taking the derivatives on both sides and seeing the results are equal!)

Your team of mechanical engineers says that the materials they have built the nose cone out of have  $\lambda = 4$ , but their measurement is only accurate to within  $\pm 0.1$ . The rocket team plans to measure the temperature at the location  $x = \pi/2$  as a function of time. However, because the rocket is vibrating during re-entry, they can only measure position to within  $\pm 0.05$ .

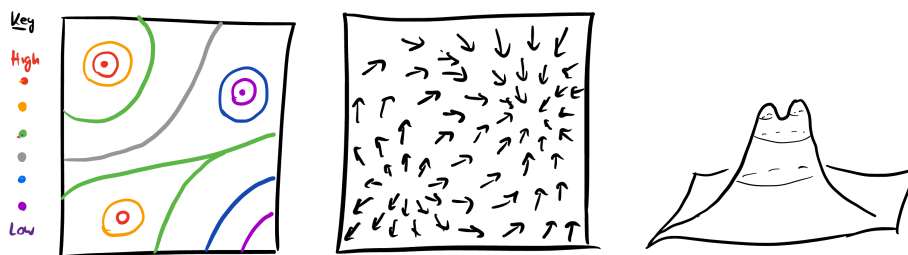
- Use *error analysis* to figure out how far off the actual temperature reading might be from the predicted theoretical value by  $H(x, t)$  at  $t = 0$ .

$$dH = H_x dx + H_\lambda d\lambda$$

### Level Sets & The Gradient

Below are depictions of three functions  $f, g, h : \mathbb{R}^2 \rightarrow \mathbb{R}$ . In the first, I have given you a picture of the level curves of  $f(x, y)$ . In the second, I have drawn the gradient of  $g$ . In the third, I have drawn a *graph* of  $h$ . Your job is to practice *translating* between these three modalities of understanding a multivariate function.

- Draw a sketch of the gradient of  $f$ , and explain in a sentence or two your diagram: why did you draw the arrows you did?
- Draw a sketch of the level sets of  $g$ , and explain in a sentence or two your diagram: why did you draw the curves you did?
- For  $h$ , you can choose: either draw me a diagram of the level sets of  $h$ , or the gradient of  $h$ . Explain in a sentence why your sketch looks like it does.



### Directional Differentiation

A small bug is flying through the air in a room, where the temperature distribution (in degrees Celsius) is given by a scalar function  $T : \mathbb{R}^3 \rightarrow \mathbb{R}$

$$T(x, y, z) = xy + yz + xyz - (x^2 + y^2)$$

The bug's motion is modeled by a parametric curve  $\vec{c} : \mathbb{R} \rightarrow \mathbb{R}^3$ :

$$\vec{c}(t) = \left( 1 + t, \frac{t^2}{2} - 1, \frac{1}{3} + t + \frac{t^3}{3} \right)$$

## Assignment 6

- What is the temperature where the fly is at  $t = 2$ ?
- How quickly is the temperature changing in the direction of the fly's motion at  $t = 2$ ?

The fly is seeking out a plate of warm food in the room to land on, and evolution has endowed the fly with a simple strategy that works: at a given point  $\vec{p}$  in space, the fly should attempt to turn and fly towards the nearest warm object (that is, in the direction in which temperature is increasing the quickest).

- From its position at  $t = 2$ , what (unit vector) direction should the fly head to approach the food most efficiently?
- What is the angle between the current fly's flightpath and this new optimal direction?

## Solutions

### Heatshields

The heat equation describes how a function measuring the heat of an object varies in both  $x$  and  $t$ . It sets the first time derivative of  $H$  equal to (a positive multiple of) the second space derivative of  $H$ . First derivatives measure the rate of change, and second derivatives measure *concavity*, so this equation says

The rate of change of  $H$  in time is proportional to the concavity of  $H$  in the  $x$  direction.

We can use this to understand qualitatively what happens over time: where a function's concavity is positive, the equation says  $\partial_t H$  is positive, so  $H$  is *increasing*. Similarly, if the concavity is negative, then  $H$  must be *decreasing*. Looking at the plot, this means the *peaks* (maxima, so negative concavity) are decreasing and the minima are increasing, so the temperature is evening out over time.

For the error analysis, we need to compute  $H_x$  and  $H_\lambda$ :

$$H_x = \frac{\partial}{\partial x} e^{-\lambda^2 t} \sin(\lambda x) = e^{-\lambda^2 t} \lambda \cos(\lambda x)$$

Now we need to evaluate this at the point we are interested in:  $\lambda = 4$ ,  $x = \pi/2$ , and  $t = 0$ :

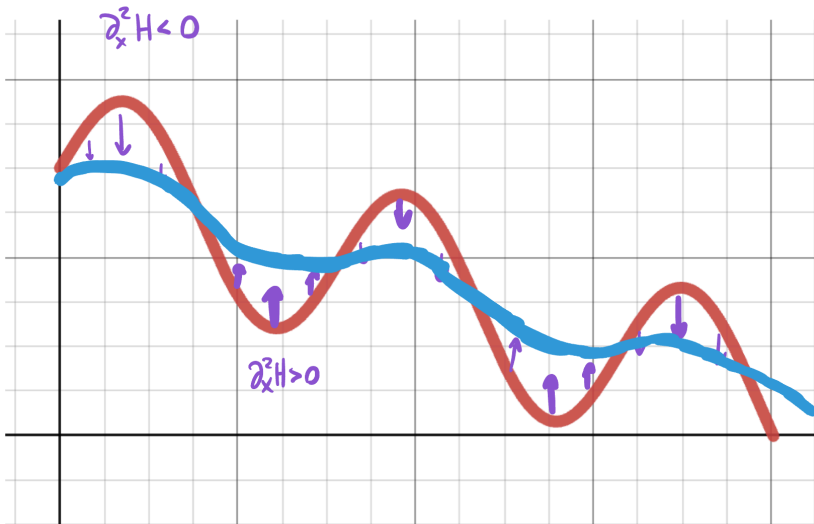


Figure 17.33.: The heat equation causes  $H$  to even out over time.

$$H_x = e^{4^2 \cdot 0} \cdot 4 \cdot \cos\left(4 \frac{\pi}{2}\right) = 1 \cdot 4 \cdot 1 = 4$$

Similarly we need to find  $H_\lambda$  and evaluate at the values of interest. Since  $\lambda$  shows up twice in our formula, we are going to need to use the product rule:

$$H_\lambda = \frac{\partial}{\partial \lambda} e^{-\lambda^2 t} \sin(\lambda x) = -2\lambda e^{-\lambda^2 t} \sin(\lambda x) + e^{-\lambda^2 t} x \cos(\lambda x)$$

$$\begin{aligned} H_\lambda &= -2 \cdot 4 \cdot e^{-4 \cdot 0^2} \sin\left(0 \cdot \frac{\pi}{2}\right) + e^{-4 \cdot 0^2} \cdot \frac{\pi}{2} \cdot \cos\left(4 \frac{\pi}{2}\right) \\ &= -2 \cdot 4 \cdot 1 \cdot 0 + 1 \cdot \frac{\pi}{2} \cdot 1 \\ &= \frac{\pi}{2} \end{aligned}$$

Thus we now know how to estimate the potential error  $dH$  in a heat measurement caused by our uncertainty in the material parameter  $\lambda$  and the sensor position  $x$ :

## Assignment 6

$$\begin{aligned}dH &= H_x dx + H_\lambda d\lambda \\ &= 4dx + \frac{\pi}{2} d\lambda \\ &= 4(0.05) + \frac{\pi}{2}(0.1) \\ &= 0.35707 \dots\end{aligned}$$

Thus, if our measurement is more than 0.36 or so away from the theoretical prediction, we know there's an error in our experiment or theory somewhere!

### Level Sets & The Gradient

In this problem the key is to remember the following facts about the gradient:

- The gradient points in the direction of steepest increase of the function
- The gradient is orthogonal to level sets of the function

And the following facts about level sets of a function:

- Near a max or a min, level sets look like concentric circles
- Near a saddle, level sets make an "X" shape.

In the first part of the problem, we are given the level sets and asked to draw the gradient. We automatically know the gradient vectors will be perpendicular to the level sets, and since the level sets come with a legend telling us which ones are high valued and which ones are low, we can see the direction. I started by drawing a couple gradient vectors directly on the level set sketch, and used these as a guide to fill in the rest:

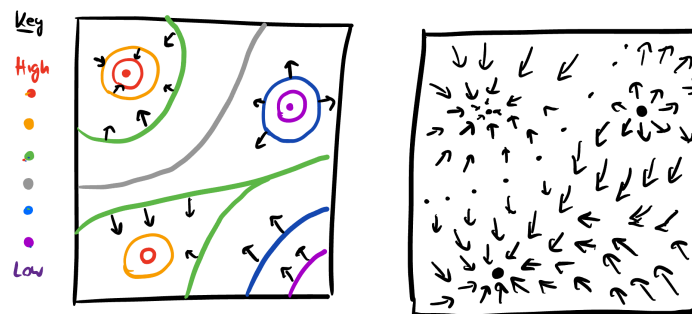


Figure 17.34.: Drawing the gradient from level sets



Next we started with a plot of the gradient and needed to draw level sets. First we notice there is one spot where the gradient is pointed inwards, and one spot where its pointed outwards in every direction. Since the gradient always points towards the steepest increase, these are a max and min respectively. And, we know maxes and mins are surrounded by concentric circle level sets. Second, we can see the gradients all point in approximately the same direction across the middle of the image, so the level set there is approximately linear. From this, we can fill in the rest by making sure they never cross (making a saddle) or close up to form any other families of circles (making a new max or min):

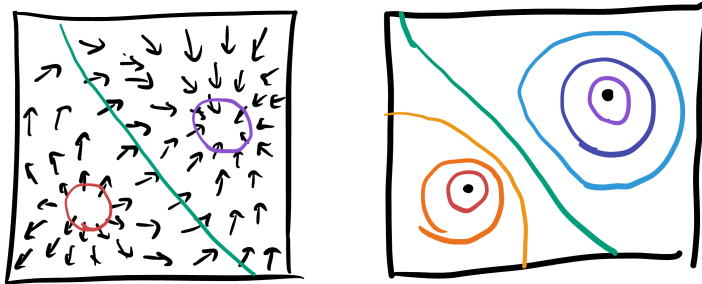


Figure 17.35.: Drawing level sets given the gradient.

Finally we are given a picture of a graph in 3D and told to draw its gradient or its level sets. I'll draw the level sets: if you want to draw the gradient you can follow the procedure we did above to convert my picture. The first thing we notice is the function graphed has two maxes and a saddle. So, let's label those and then draw what the nearby level sets look like:

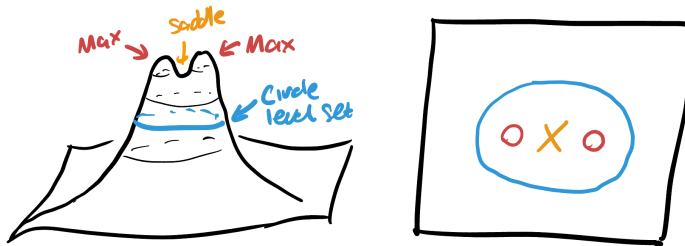


Figure 17.36.: The local level sets near critical points provide a very helpful guide.

We also see by looking farther down the function that the level sets must eventually become circles, so we add one of these too. Then we can just fill everything in, by making sure the level sets never cross or close off to form more families of circles:

## Assignment 6

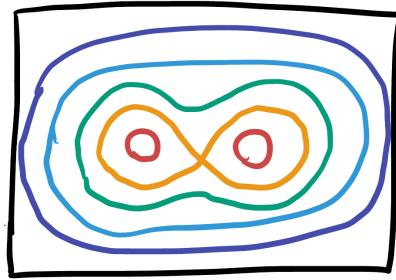


Figure 17.37.: The finished level set plot.

### Directional Differentiation

At  $t = 2$  we can find the fly's location by just plugging in 2 to the parametric curve:

$$\vec{c}(2) = \left( 1 + 2, \frac{2^2}{2} - 1, \frac{1}{3} + 2 + \frac{2^3}{3} \right) = (3, 1, 5)$$

To find the temperature here, we then plug this into  $T(x, y, z)$ :

$$T(\vec{c}(2)) = T(3, 1, 5) = (3)(1) + (1)(5) + (3)(1)(5) - (1^2 + 3^2) = 13$$

To find how quickly the temperature is changing in the direction of the fly's motion, we first need to find this direction. That means finding the unit vector in the direction of  $\vec{c}'(2)$ .

$$\vec{c}'(t) = \langle 1, t, 1 + t^2 \rangle$$

$$\vec{c}'(2) = \langle 1, 2, 5 \rangle$$

$$\|\vec{c}'(2)\| = \|\langle 1, 2, 5 \rangle\| = \sqrt{1^2 + 2^2 + 5^2} = \sqrt{30}$$

Thus the unit vector in the direction of the fly's motion is

$$v = \left\langle \frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}} \right\rangle$$

To find the rate temperature is changing in this direction, we need to find the *directional derivative*. This is computed using the gradient vector

$$D_v T = \nabla T \cdot v$$

So we next need to find the gradient at the fly's position:

$$\nabla T = \langle y + yz - 2x, x + z + xz - 2y, y + xy \rangle$$

And evaluating at the fly's position (3, 1, 5) gives

$$\nabla T(3, 1, 5) = \langle 1 + 5 - 6, 3 + 5 + 15 - 2, 1 + 3 \rangle = \langle 0, 21, 4 \rangle$$

Dotting this with the direction vector gives the directional derivative we seek:

$$\langle 0, 21, 4 \rangle \cdot \left\langle \frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}} \right\rangle = \frac{42}{\sqrt{30}} + \frac{20}{\sqrt{30}} = \frac{62}{\sqrt{30}} \approx 11.32$$

**Thus, the rate of change of temperature is 11.32 degrees per second**

Next, the direction  $\vec{d}$  where temperature is increasing fastest is the direction of the gradient, so we just need to divide  $\nabla T$  by its magnitude:

$$\|\nabla T\| = \|\langle 0, 21, 4 \rangle\| = \sqrt{457}$$

$$\vec{d} = \left\langle 0, \frac{21}{\sqrt{457}}, \frac{4}{\sqrt{457}} \right\rangle$$

And finally, to find the angle the fly should turn to head in this direction, we can use the dot product! We already have the unit vector  $v$  in the fly's direction, and the unit vector  $d$  in the direction of fastest increase in temperature. So

$$\cos \theta = v \cdot d = \frac{42}{\sqrt{30}\sqrt{457}} + \frac{20}{\sqrt{30}\sqrt{457}} = \frac{62}{\sqrt{13710}} \approx 0.529$$

**Taking the inverse cosine, we find that  $\theta = 1.0127$  radians, or approximately 58 degrees.**



# Assignment 7

## Problems

### Function Sketching

Consider the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by the following formula:

$$f(x, y) = x^3 + y^3 - 3xy + 14$$

In this problem, we work to understand  $f$  without any technology: to showcase the power that calculus has brought us.

- Find the critical points of  $f$ , and classify these as max, mins or saddles via the second derivative test.
- Use this classification to sketch the level curves of  $f$  like we did in class. Start by drawing the level curves near each max, min and saddle. Then use that level curves cannot cross (away from a saddle) and there are no other max/mins to fill in the rest of the picture.
- Use this sketch of the level curves of  $f$  and the information about which points are maxima and minima to sketch the *gradient* of  $f$ .

### Optimization

You are once again working on the engineering team developing new rocket engines for the Artemis mission, where on a previous homework you submitted recommendations to your team members on how to modify the nozzle width and fuel temperature to increase efficiency. Now, you are tasked with taking this efficient design and trying to maximize the thrust during the initial burn to leave earth's atmosphere, which is where most fuel gets used up.

Having already set the nozzle diameter and fuel temperature, you look to study the effect of two other variables: the overall radius of the rocket and the slope of the nose-cone (both of which greatly affect air resistance) on the net thrust. Your colleagues run a

### Assignment 7

computational simulation of the atmospheric physics, and determine that the net force varies as a function of the radius  $r$  and slope  $s$  as

$$F(r, s) = 8 + 4rs - 4r^4 - 4s^4$$

Find the critical points of  $F$ : these are the configurations that you and your engineering team should investigate more closely.

After figuring this out, what radius and slope should you recommend to the manufacturing team? (Remember radius should be a positive number, as its a length).

## Constrained Optimization

On the last problem, you helped an engineering team for the NASA Artemis moon mission optimize the width and slope of their rocket's nose cone for aerodynamic efficiency.

However, after a brief celebration you receive news back from the manufacturing team: your solution simply will not work! While it certainly is the most efficient with regard to air friction, the nose cone is but one piece of a very large and complicated rocket, and there's other constraints that need to be taken into account.

Indeed, there is a fixed cap on the *total weight* of the rocket overall, and a fixed percentage of that can be used on materials in the manufacture of the nose cone. You are allotted exactly 800kg of material from which to design the nose cone, and asked to produce a design weighing exactly this much (so that other teams know ahead of time the nose cone's final weight, and can plan accordingly).

The original nosecone design violates this, and indeed the team at NASA suggests that you throw out the entire model entirely and re-design the nosecone using lighter-weight specifications. Your new model has a different dependency of thrust on radius and efficiency:

$$F(r, s) = 4 - r^2 - s^2 + 3rs$$

With this new model, you go back to your modeling software and ask the computer to estimate the weight of a nose cone with radius  $r$  and slope  $s$ : it tells you that the weight can be modeled by

$$W(r, s) = 100(s^2 + r^2)$$

(where we require that  $r > 0$  and  $s \geq 0$  as they are a distance and a slope)

Given this thrust equation and weight constraint, what should you bring back to the manufacturing team as the best  $(r, s)$  parameters for aerodynamic flight?

## Solutions

### Function Sketching

To sketch the level sets and gradient of this function, we first must find the critical points and classify them. Taking the gradient,

$$\nabla f = \langle 3x^2 - 3y, 3y^2 - 3x \rangle$$

So the system of equations we need to solve is (after dividing by the 3):

$$\{x^2 = y, y^2 = x\}$$

Substituting the first equation into the second we see

$$y^2 = (x^2)^2 = x^4 = x$$

Thus either  $x = 0$  (and hence  $y = 0$ ) or  $x^3 = 1$ . But this only has the solution  $x = 1$  (and hence  $y = 1$ ), so there are two critical points

$$\{(0, 0), (1, 1)\}$$

To classify them, we need to apply the second derivative test:

$$f_{xx} = 6x \quad f_{xy} = -3 \quad f_{yy} = 6y$$

Thus the Hessian matrix and its determinant are

$$H = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix} \quad D = \det H = 36xy - 9$$

### Assignment 7

Evaluating this at the critical points gives  $D(0, 0) = -9$ , so  $(0, 0)$  is a saddle, and  $D(1, 1) = 36 - 9 = 25$ , so  $(1, 1)$  is either a max or min. To figure out we look at the sign of  $f_{xx}$  or  $f_{yy}$ . Both of these are positive, so its a minimum.

To sketch the level sets, we can start with the local model of a minimum (concentric rings) at  $(1, 1)$  and of a saddle (an X shape) at  $(0, 0)$ , then fill in the rest of the plot making sure no other level sets cross or close up into circles.

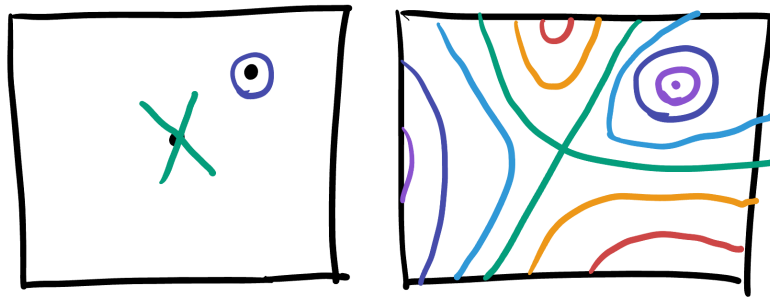


Figure 17.38.: Drawing level sets from information about critical points.

Given this picture, we can draw the gradient by remembering its orthogonal to level sets, and points in the direction of maximum increase. Since we know that saddles go upwards along one axis and downwards along the other, knowing where the minimum is sorts out the directions of everything else:

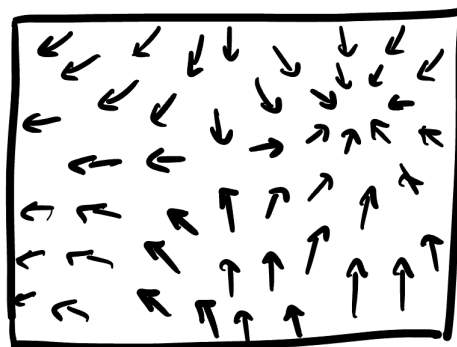


Figure 17.39.: Drawing the gradient from the level sets.



## Optimization

To optimize the behavior of this rocket, we are searching for maxima in  $(r, s)$ . That means we again must begin by finding critical points and classifying them. Starting with the gradient

$$\nabla F = \langle 4s - 16r^3, 4r - 16s^3 \rangle$$

As the critical points are where the gradient vanishes, this leads to the system of equations

$$\{s = 4r^3, r = 4s^3\}$$

Similar to the previous problem, we can substitute one of these into the other to get an equation in a single variable:

$$s = 4r^3 = 4(4s^3)^3 = 4^4s^9$$

Thus, either  $s = 0$  (and hence  $r = 0$  too) or we can divide by  $s$  and see that

$$1 = 4^4s^8 \implies \frac{1}{4^4} = s^8$$

Taking the fourth root of both sides gives

$$s^2 = \frac{1}{4} \implies s = \pm \frac{1}{2}$$

Plugging these into the equation  $r = 4s^3$  to get  $r$  in each case, we find three critical points:

$$\left\{ (0, 0), \left(\frac{1}{2}, \frac{1}{2}\right), \left(\frac{-1}{2}, \frac{-1}{2}\right) \right\}$$

Now to classify these we must apply the second derivative test:

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix} = \begin{pmatrix} -48r^2 & 4 \\ 4 & -48s^2 \end{pmatrix}$$

$$D = \det H = (-48r^2)(-48s^2) - (4)(4) = 48^2r^2s^2 - 4^2$$

### Assignment 7

Plugging in the critical points, we find that

$$D(0, 0) = -16 \quad D\left(\frac{1}{2}, \frac{1}{2}\right) = 144 \quad D\left(\frac{-1}{2}, \frac{-1}{2}\right) = 144$$

Thus  $(0, 0)$  is a saddle, and we need to look closer at the sign of  $f_{rr}$  to determine the other two extrema. In each case  $f_{rr} = -12 > 0$  so they are each maxima.

BUT: remember that  $r$  is a radius in the problem, and must be a positive number, so we can actually ignore this critical point - it doesn't correspond to a buildable rocket. This means we should suggest the parameters

$$(r, s) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

to NASA, where the maximal value is

$$F(1/2, 1/2) = \frac{31}{4}$$

### Constrained Optimization

To solve the constrained optimization problem, we need to employ the method of Lagrange multipliers. We need to find the gradient of the efficiency function as well as of the constraint.

$$\nabla F = \langle -2r + 3s, -2s + 3r \rangle$$

The constraint on weight is  $100(s^2 + r^2) = 800$ , so we can simplify the numbers a bit by first dividing by 100 to get  $s^2 + r^2 = 8$ . Now,  $\nabla W = \langle 2r, 2s \rangle$  and so the system of equations to be solved is

$$\nabla F = \lambda \nabla W$$

$$\begin{pmatrix} 3s - 2r \\ 3r - 2s \end{pmatrix} = \begin{pmatrix} 2\lambda r \\ 2\lambda s \end{pmatrix}$$

One solution to this system is  $r = 0$  and  $s = 0$  as then both equations just become  $0 = 0$  which is true. When they are nonzero, we can solve each equation for  $2\lambda$  by dividing by a variable and set them equal:

$$\frac{3s - 2r}{r} = 2\lambda = \frac{3r - 2s}{s}$$

This eliminates  $\lambda$  which is good - we don't care about its value anyway - and gives us an equation just in terms of  $r$  and  $s$ . Multiplying through by the denominators we can simplify a bit

$$(3s - r)s = (3r - s)r$$

$$3s^2 - rs = 3r^2 - rs$$

$$3r^2 = 3s^2$$

$$r^2 = s^2$$

Taking the square root of both sides we find  $r = \pm s$ , which we can then use in the constraint. In either of these cases,

$$r^2 + s^2 = r^2 + r^2 = 2r^2 = 8 \implies r^2 = 4 \implies r = \pm 2$$

This gives us four more critical points: when  $r = 2$  we can have  $s = \pm 2$  so we get  $(2, 2)$  and  $(2, -2)$ , similarly when  $r = -2$ , where we get  $(-2, -2)$  and  $(-2, 2)$ .

Thus, the five critical points for this system are

$$\{(0, 0), (2, 2), (2, -2), (-2, 2), (-2, -2)\}$$

But - not all of these are physically relevant to the problem! We need both  $r > 0$  and  $s \geq 0$  for the model to make sense, and only one of these satisfies both of these constraints:

$$(r, s) = (2, 2)$$

The new efficiency of this model is

$$F(2, 2) = 4 - (2)^2 - (2)^2 + 3(2)(2) = 4 - 4 - 4 + 12 = 8$$



# Study Guide II

Write a *study guide* for yourself in preparation for Midterm I.

**You can write the guide however helps you most**, but here are some possible suggestions:

- Write down the important definitions for each topic you want to review. Explain them in your own words!
- Either come up with an example problem (or find one on WebAssign) for each topic you want to review, and write out its solution by hand! You'll be handwriting solutions on the exam, so this is a good way to get yourself in that mindset and away from online work.
- For each topic you want to review, write to yourself how confident you are with it (maybe on like a 1-10 scale, if that's helpful to you): this way when you're studying you can look back and easily remember which things you were struggling with.
- If there's multiple ways to do something, make a note of that for yourself!

The second midterm will focus mostly on the chapter *Multivariate Differentiation*, and the beginnings of *Multivariate Integration* (Only the first section, *double integrals*). Here's a short cliffnotes of what we have covered (essentially a table of contents of the notes above.)

## Differentiation

### Functions

- Understanding graphs of multivariable functions.
- Understanding level sets of multivariable functions.

## Partial Derivatives

- Know the definition and conceptual meaning of partial derivatives.
- Be able to use partial derivatives to tell when a function is increasing / decreasing / concave up /concave down in a coordinate direction.
- Computing implicit partial derivatives, and using them to find slopes.
- Be able to compute higher order partial derivatives.
- Being able to express in words what a partial differential equation says. Understanding qualitative behavior of partial differential equations.

## Linearization

- Find implicit and parametric tangent planes to graphs  $z = f(x, y)$ .
- Find normal vectors to graphs of functions  $z = f(x, y)$ .
- Use differentials to estimate the value of a function near a point where you can compute it.
- Use differentials for error analysis: quantification of error in measurements.

## The Gradient

- Know the definition of the gradient, how to compute it.
- Be able to use the gradient to compute directional derivatives.
- Know the geometry of the gradient: specifically, the meanings of its direction and magnitude.
- Know the relationship between gradients of a function and its level sets.

## Extrema

- Be able to find the critical points of a function
- Be able to classify the critical points of a function as maxes mins or saddles.
- Given information about maxes mins and saddles from such a computation, be able to sketch the level sets of a function.

## Optimization

- Understand how to deal with constraints via substitution.
- Know the concepts behind the technique of Lagrange multipliers

- Be able to use the technique of Lagrange multipliers to find the maxima and minima of a function along a constraint.
- Be able to combine Lagrange multipliers with the first and second derivative test of the previous section to solve constrained optimization problems with inequalities.

## **Integration**

### **Double Integrals**

- Be able to compute the double integral of a function over a rectangle.
- Be able to compute an iterated integral with variable bounds.
- Be able to change the order of integration on a double integral.





# Assignment 8

## Problems

### The Mass of Steel Sheets

Integrals measure the *area* under a curve, but appear in a much wider context than simply geometry: often this area is encoding the *total amount* of some physical property distributed along an interval, rectangle or surface.

One common example is mass: if you have an object of a given fixed *density*  $\rho$  and volume  $V$ , its possible to find the mass as just the density times volume. But if you have an object whose density varies, then the total mass is the *integral* of its density! This may be familiar already in one dimension from Calc I/II or introductory physics/engineering, where the mass of a 1-dimensional rod from  $x = a$  to  $x = b$  is given by

$$m = \int_a^b \rho(x) dx$$

Multivariate integrals bring these techniques much closer to real world application, as real-world objects are often much more than lines! In this problem we will learn how to calculate the mass of a corrugated steel sheet, the kind used in many manufacturing and engineering designs due to its good ratio of strength to weight.



Figure 17.40.: A sheet made of corrugated steel commonly used in construction.

The density of a flat steel sheet is constant (at 2mm thick, its 15kg per square meter). However, a corrugated sheet has a non-constant density across its surface, due to the

### Assignment 8

fact that it bends in one direction (and there is more sheet, hence more steel, in the bends). The density (in kilograms / square meter) of a corrugated sheet is well modeled by the simple expression

$$\rho(x, y) = 15 + 0.5 \sin^2(5\pi x)$$

(The variable  $y$  does not show up in the formula as the sheet is only corrugated in one direction).



Figure 17.41.: A building made of corrugated steel plates.

You are on a construction team that is looking to plate the outside of a new building in corrugated steel sheets. You are told that a single panel of such sheeting is a rectangle running from  $x = 0$  to  $x = 3$  meters, and  $y = 0$  to  $y = 5$  meters.

- Set up a double integral that gives the mass of this sheet. How much does it weigh?
- The building you are looking to cover in panels has an outside surface area of 800 square meters. How many panels do you need to order to complete the project? If each truck bringing in panels has a maximum capacity of one ton (1,000kg) how many truck loads will be required on construction day?

### Building Roads

You are back working in the city planners office, this time consulting on a new road construction project. The planned road is to follow a relatively straight path through a hilly terrain, and to prepare for construction you have recommended bringing in a team of steam shovels and bulldozers to cut a path into the hillside so the road can run straight.

You consult a GPS company to get the elevation data for your county, and learn that the area through which the road passes has a height (in meters)  $h(x, y)$  above the bedrock:

$$h(x, y) = 0.01xy + 0.001x^2$$



Figure 17.42.: A road cut through a hillside is built by dynamiting the rock along a straight path and removing it, leaving a level surface for pavement.

In these coordinates the road is to be approximately 6 meters wide, centered on the path  $y = 3x - 1$ . That is, the lower edge of the road is described by  $y = 3x - 4$  and the upper edge of the road is described by  $y = 3x + 2$ . You are responsible for the construction of this road between its start at  $x = 0$  and its first intersection 1 kilometer to the east, at  $x = 1000$ .

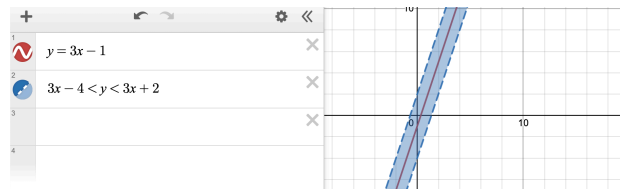


Figure 17.43.: The path of the proposed road in the  $xy$  plane: it goes from  $x = 0$  to  $x = 1000$  and lies between the curves  $y = 3x - 4$  and  $y = 3x + 2$ .

The main thing the city needs to know is how much earth needs to be removed during construction. They own a fleet of dumptrucks, each of which can carry  $100m^3$  of dirt at a time, but they would like an estimate of the total number of trips this will take, as that will help plan an accurate timeline for the project.

- Set up a double integral that measures the total amount of land that needs to be removed from the current site of the road.
- Evaluate this double integral. How many truckloads should the city plan for?

## Calculating Air Pressure

Our atmosphere - the thin layer of air surrounding our planet making all life possible - is held to the surface via gravity. There is no barrier holding the atmosphere in, but instead it thins out rapidly from its density at the surface to a near perfect vacuum up

### Assignment 8

in space. Using some classical physics, one learns that in fact the atmospheric density decreases exponentially, following the formula

$$\rho(x, y, z) = \rho_0 e^{-z/H}$$

Where  $\rho_0$  is the density of air at sea level and  $H$  is the *characteristic height* of the atmosphere: a number controlling how quickly the atmosphere thins out with altitude. These numbers are different on each planet with an atmosphere: on Earth we find that  $\rho_0 = 0.0752$  pounds per cubic foot, and  $H = 6.21$  miles.

Usually, we can happily ignore air pressure in day-to-day life, as it presses on all sides of us equally, and our bodies have evolved to support this pressure. However, air pressure is very important to take into account in certain engineering situations: especially when designing *vacuum chambers*. A vacuum chamber is an enclosure (ranging from a small box, to an entire room) from which all of the air can be pumped out for experiments requiring an air-free environment (such as testing rocket engine parts).

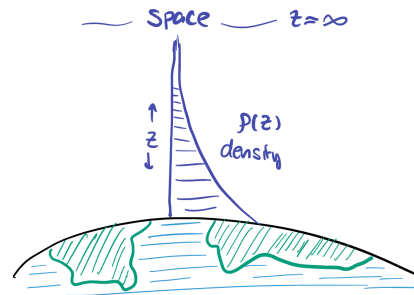


Figure 17.44.: The atmosphere drops off exponentially in density, and its mass is calculated as a triple integral of its density over the entire air column, from  $z = 0$  to  $z = \infty$  over an area  $R$  in the  $xy$  plane.

As part of your job back at the rocket factory, you are tasked with helping the engineering team design a new vacuum chamber. This chamber is to be a perfect cube, 10 feet on each side.

Specifically we can model the top face of this vacuum chamber as a square region  $R$  in the plane with  $0 \leq x \leq 10$  and  $0 \leq y \leq 10$ .

Above this square  $R$ , there is a column of air extending all the way into space: we denote this region as

$$E = \{(x, y, z) \mid (x, y) \in R, z \geq 0\}$$

To calculate the weight of the atmosphere pressing down on the vacuum chamber, we need to do a *triple integral* of the density of the atmosphere inside of this column of air:

$$M = \iiint_E \rho(x, y, z) dV$$

How much weight is pressing down on the top of the vacuum chamber from the air?  
*Hint: the range of  $z$  in the region  $E$  will go all the way into space, so you'll end up with an improper integral from Calc II. It's easiest to calculate things leaving  $\rho_0$  and  $H$  as constants, and only plugging in their numerical values at the end: remember to convert everything to the same units (say, pounds and feet!)*

## Solutions

### Mass of Metal Sheets

We are given the mass density  $\rho(x, y) = 15 + 0.5 \sin^2(5\pi x)$  and the size of the sheet  $x \in [0, 3], y \in [0, 5]$ . So, finding its mass just amounts to calculating a double integral:

$$M = \iint_R \rho dA = \int_0^5 \int_0^3 \left(15 + \frac{1}{2} \sin^2(5\pi x)\right) dx dy$$

We evaluate this as an iterated integral, so we can first focus just on the  $x$  integral. This has  $\sin^2$  in it, so we use a trigonometric identity to simplify

$$\begin{aligned} \sin^2(5\pi x) &= \frac{1 - \cos 10\pi x}{2} \\ \int_0^3 \left(15 + \frac{1}{2} \frac{1 - \cos(10\pi x)}{2}\right) dx &= \int_0^3 15 dx + \int_0^3 \frac{1}{4} (1 - \cos(10\pi x)) dx \\ &= 45 + \frac{1}{4} \left( x - \frac{\sin(10\pi x)}{10\pi} \right) \Big|_0^3 \\ &= 45 + \frac{1}{4} \left( 3 - \frac{0}{10\pi} \right) \\ &= 45 + \frac{3}{4} = 45.75 \end{aligned}$$

Now, we integrate the result with respect to  $y$ :

### Assignment 8

$$M = \int_0^5 45.75 dy = 45.75 * 5 = 228.75 \text{kg}$$

The building has a surface area of  $800m^2$  and each panel is  $3 \times 5 = 15m^2$ . Thus we need  $800/15 = 53.33 \dots$  panels, so in reality we need 54 panels to make sure we can cover it all.

Each truck can carry 1000kg total which is

$$\frac{1000}{228.75} = 4.37 \text{ panels}$$

Thus, every truck can carry 4 panels (as 5 would be too many for its weight). We need 54 panels total, so the number of truck loads required is

$$\frac{54}{4} = 13$$

### Building Roads

The amount of earth to be cleared away is the double integral of the height of the ground  $h(x, y)$  over the region defined by the future road. The problem gives us  $h$ , and describes the region as

$$R = \{0 \leq x \leq 1000, 3x - 4 \leq y \leq 3x + 2\}$$

Thus, the integral which determines the volume of rock to be cleared is

$$V = \iint_R h dA = \int_0^{1000} \int_{3x-4}^{3x+2} 0.01xy + 0.001x^2 dy dx$$

We compute this as an iterated integral, focusing first only on the innermost integral:

$$\begin{aligned} & \int_{3x-4}^{3x+2} \frac{1}{100}xy + \frac{1}{1000}x^2 dy \\ &= \frac{1}{200}xy^2 + \frac{1}{2000}x^2y \Big|_{3x-4}^{3x+2} \\ &= \frac{1}{200}x((3x+2)^2 - (3x-4)^2) + \frac{1}{2000}x^2((3x+2) - (3x-4)) \end{aligned}$$

This can be simplified with some algebra:

$$\begin{aligned}
 &= \frac{1}{200}x(36x - 12) + \frac{1}{2000}x^2(6) \\
 &= \frac{12}{200}(3x^2 - x) + \frac{6}{2000}x^2 \\
 &= \frac{183}{1000}x^2 - \frac{3}{50}x = 0.189x^2 - 0.06x
 \end{aligned}$$

Next, we integrate with respect to  $x$ :

$$\begin{aligned}
 &\int_0^{1000} 0.189x^2 - 0.06x \, dx \\
 &= 0.183 \frac{x^3}{3} - 0.06 \frac{x^2}{2} \Big|_0^{1000} \\
 &= 0.061(1000)^3 - 0.03(1000)^2 \\
 &\quad 60,970,000 \, m^3
 \end{aligned}$$

Since each truckload can only carry 1000 cubic meters, the city should plan for 60,970 truckloads! No wonder road construction takes so long.

### Calculating Air Pressure

We are looking to calculate the air pressure of the atmosphere on the square region

$$R = \{0 \leq x \leq 10, 0 \leq y \leq 10\}$$

That means we need to calculate the *mass of the entire atmosphere* over this region. Approximating the atmosphere as extending all the way up into space (to infinity) we can write this region as

$$E = \{(x, y, z) \mid (x, y) \in R, 0 \leq z \leq \infty\}$$

To find the mass we need to perform an integral over  $E$  of the *density* of the atmosphere, which we were given  $\rho(x, y, z) = \rho_0 e^{-z/H}$ .

We can express the triple integral as an iterated integral given our bounds

Assignment 8

$$M = \iiint_E \rho \, dV = \int_0^{10} \int_0^{10} \int_0^{\infty} \rho_0 e^{-z/H} \, dz \, dy \, dx$$

This allows us to just focus on the  $z$ -integral first: remembering that  $\rho_0$  and  $H$  are just constants,

$$\begin{aligned} & \int_0^{\infty} \rho_0 e^{-z/H} \, dz \\ &= \rho_0 e^{-z/H} \frac{1}{-1/H} \Big|_0^{\infty} \\ &= -H \rho_0 e^{-z/H} \Big|_0^{\infty} \\ &= -H \rho_0 (0 - 1) = H \rho_0 \end{aligned}$$

Now we just need to integrate over  $x$  and  $y$ , which is easy since  $H$  and  $\rho_0$  are constant:

$$\int_0^{10} \int_0^{10} H \rho_0 \, dy \, dx = H \rho_0 \int_0^{10} \int_0^{10} \, dy \, dx = 100 H \rho_0$$

We have the values of the constants:  $\rho_0 = 0.0752$  pounds per cubic foot, and  $H = 6.21$  miles, but to use them we need to make sure everything is in the same units. Our integral  $x, y, z$  were all in feet, and so is the constant  $\rho_0$ , so that's good. But  $H$  was given in miles, so we need to *convert it to feet*:

$$H = 6.21 \text{ miles} \cdot \frac{5280 \text{ feet}}{\text{mile}} = 32788.8$$

Plugging these two constants in, we find that the weight of the atmosphere pushing down on our vacuum chamber is

$$100 \cdot 32,788 \cdot 0.0752 = 246,571$$

Thus we need to build things so that they can support 246,561 pounds of atmospheric pressure.



## Reflection & Correction II

The goal of a college course is always *learning* - with homework, exams, and other assignments all just vehicles (and motivation) to help learning occur. Thus, my main goal from this exam is to help you all *learn* where you currently stand in the course, and to set yourself up for success going forward.

- The first part of this process was the *studying for the exam*: where you reviewed all the material on vectors and curves. Here you could earn extra points towards your grade (if you want to use the hold-me-accountable-grading) by **writing a study guide for yourself**.
- The second part of the process was *taking the exam itself*. Here the goal is for both you and me to learn what you know so far in the course, and gauge what needs to be done going forward.

Now, we are at the *post-exam-period* where we try to use the test to learn from our pasts. To really get set up on the right path requires some work, so there will be a *two opportunities to earn points* here as motivation.

- First - you can do an **extra credit exam correction**. You will be able to regain the *half the amount of points missed on one problem of your choice*. That is - if you got a 1/5 on one problem you missed 4 points, and so you can earn back 2 points by doing a correction. (Note this is *half* the extra credit compared to the first exam: this fits into a pattern of full EC on exam 1, half the EC on exam 2, and no EC on the final, as the course is over).
- The second, is to **write an exam reflection** as part of the hold-me-accountable grading system. Remember this grading system gives weight to other assignments like this, and in the process lowers the amount that exams count towards your final grade.

### Extra Credit: Exam Correction

**For HALF of the missed points back any problem of your choice:** Write a *study guide* for the problem, addressed to your past self, (or your future self, when you review

for the final). **MAKE SURE TO INCLUDE BOTH THE PROBLEM STATEMENT, AND THE ORIGINAL SCORE YOU RECEIVED**

You are not simply fixing the mistakes in your old solution, but writing a rather involved document *teaching the mathematics* necessary to succeed at questions like this. Your submission should be *neatly hand written* or *typed* and in full paragraphs with complete sentences. It should not be a rough draft, or an outline (bullet-point list of thoughts, etc). **Complete submissions will likely be several pages in length.** Below is an outline to help you structure such a lesson.

**Analysis of your original solution** What is the complete statement of the problem you are writing a study guide for? When you were working on the exam, what did you think about or try? If you were stuck at the beginning, or did not write much, what made it difficult to make progress?

**Teach the necessary techniques** Before you begin writing up your correction, learn how to solve the problem in its entirety. Look at your solution, and identify the main tools you needed: do you need partial derivatives? Parametric curves? Dot and cross products of vectors? Were some of your struggles caused by material prior to our course? (Differentiation rules, or algebraic manipulation rules?). Were your difficulties conceptual (had trouble picturing the difference between a vector and scalar function, unsure how to tell if something is a circle or cylinder) or computational (forgot the formula for cross product, etc)?

**For each mathematical technique that is integral to the solution of this problem** write a subsection reviewing this technique. Your section should include

- The technique itself (the formula / identity, or set of rules used)
- A discussion of what this technique is *for*: when should your reader (future you) think to use it in a problem?
- An example (or two, or three) of using this technique correctly, in simple problems.
- A discussion of potential pitfalls: what are situations where using this mistake can easily lead to a mistake? (For a calc 2 example; forgetting to convert  $dx$  in  $u$ -sub, or some of the more involved conversions back to  $x$  from  $\theta$  for trigonometric substitution)

**A Full Solution of the Problem** Now that you have taught your reader all of the mathematics necessary to do this problem, write up a *complete, annotated solution*. Start by repeating the problem statement, and then talk your reader through (in sentences) what you should be thinking about at each step. *I know that I have posted the exam solutions! So I am not asking you to just write symbol for symbol what I wrote. I am asking you to really explain what is going on in the solution.* Write this as though you

are truly trying to help your future self feel confident about this problem when they are reviewing this material down the road.

**A similar problem** Now that you have become an ‘expert’ at this one problem, **make up a new example question that is similar** (as in, it uses the same techniques to solve). To make sure it is indeed similar - you should solve it after proposing it! But you do not need to include the solution in your writeup. Instead, this problem will be waiting for you to try again next time you study.

## **Hold-Me-Accountable: Exam Reflection**

Reflect on the the first part of the semester, focusing on your studying techniques, your exam performance, and suggestions to your future self. Your submission should be *neatly hand written* or *typed* and in full paragraphs with complete sentences. It should not be a rough draft, or an outline (bullet-point list of thoughts, etc). There are no wrong answers, but only submissions showing real work at introspection will receive credit: remember, this is an (optional) opportunity for you to think about what works best *for you*

**How did the exam go?** While it is still fresh on your mind, think about the exam itself. How did you do compared to how you expected to do (after studying, but before the exam itself)? After getting feedback, how did the exam go relative to how you felt after taking it? Did you do better or worse than you felt you had (after leaving the exam, but before getting feedback).

If you did well - what were the big contributions to that success for you? If you hoped to have done better, what were some factors that may have affected your performance (these include comfort level with the material, but also things like not sleeping enough the day before, or time pressure etc).

**How did you implement your proposed strategy from last time?** Last reflection, you wrote yourself a specific list of things that you wanted to try to change up your study strategies. Give yourself an honest evaluation: how well did you follow your past recommendations? Additionally, look at your past recommendations critically - did you suggest things that were reasonable, given your time/energy/bandwidth this semester? Did you suggest things that actually turned out to be helpful?

**What were your study strategies?** What did you *actually do* this time in preparation for the exam? Did you study for the exam with friends, or alone? How did you use the practice exam? Of the things you did do, what felt like it had the most payoff? Did any of your studying feel unproductive for you (as in, you put a lot of time into a particular concept or strategy, but in the end still struggled with that)?

*Reflection & Correction II*

**Where is this class as a priority for you this term?** We all only have so much time in a day, and we have so many obligations in school and in life: its important to try and accurately assess your priorities when making plans for the future - so that they can be *realistic* and *helpful*. Take time here to give an honest evaluation of your course load this term: what class are you worried most about? What class do you think will take the most work going forward? What class are you least concerned about? Use your own personal ranking here in the next step, to make a *realistic* recommendation.

**What are some recommendations for your future self?** Now that you've been through this cycle twice, combine what you have learned to give yourself a concrete plan for how you will work on this class over the last month of the term.

# Assignment 9

## Problems

### Optimizing Volume of a Wineglass

We so far have been using triple integrals to compute *the total amount of some function  $f$  over a region  $E$* . Remember that when this function  $f$  is just the constant 1, you end up with an integral of the form

$$\iiint_E dV$$

and this just computes the *volume of the region  $E$* ! This is very useful in practice, as the volumes of complicated regions can be calculated using integration techniques.

As an example here, imagine you are an industrial engineer working for a company that manufactures sets of high-end drinkware: wine glasses, champagne flutes, and so on. The designers of the new collection have come up with a *parametric model*: a single model with several tune-able parameters. Their model describes the interior of the glass by a region  $E_{a,b}$ , which can be tuned by changing two parameters  $a, b$ :

$$E_{a,b} = \{(x, y, z) \mid b(x^2 + y^2) \leq z \leq a\}$$

The units here are centimeters, so the volume of the region is given in  $\text{cm}^3$ , better known as milliliters. The goal of this parametric is to be able to define different glass shapes by choosing different relationships between the parameters.

- **Wine Glasses** are shorter and fatter, so to design a wine glass we set the width and height parameters equal,  $a = b$  and get a model that depends on a single parameter.
- **Champagne Flutes** are taller and skinnier: to design one of these, we set the height parameter  $b$  equal to three times the width parameter, so  $b = 3a$ .

If the company later wants to expand its glassware collection, it does not need to go through the process of redesigning new models from scratch, it can just change the relationship between  $a$  and  $b$  to get a new look and feel.

## Assignment 9

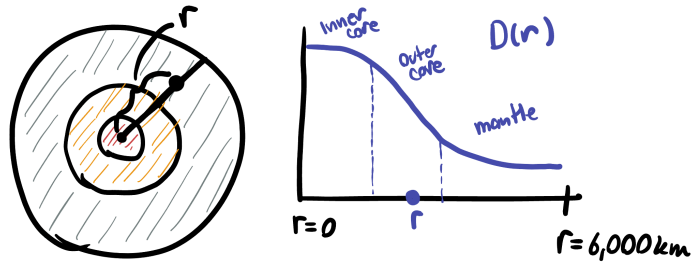


Figure 17.45.: The density of the earth varies with radius, as its composition changes from the crust to the core.

After doing a review of several cups across the industry, your manager finds the average volume of a red wine glass is 360mL. They then task you with taking the model from the design team, and constructing a wine glass as part of their collection that holds this same volume. What are the values of the parameters  $a, b$  that you need to report back to manufacturing, to have a glass of the right size?

*Hint: find the volume of  $E_{a,b}$  as a function of the parameters, by doing a triple integral in cylindrical coordinates. Use the equation given to find the bounds for  $z$  and the bounds for  $r$ ! Then, after finding the general formula, use the fact that this is to be a wine glass to get  $E_{a,b}$  in terms of a single parameter, and then solve the equation for this volume to equal 360mL using a calculator or computer.*

## The Mass of the Earth

The interior structure of the Earth is very difficult to probe: we live on the very surface of the *crust*, a rocky layer over 20 miles thick, and to date the deepest humans have ever drilled is the Kola Superdeep Borehole, dug to a depth of 7.69 miles by the USSR (and stopped when the heat and pressure was melting through all drillbits).

Thus, it is a true wonder of modern science that we are able to calculate the *density* of the earth at various depths without directly retrieving samples: instead, we use the vibrations of earthquakes (and a lot of sophisticated mathematics) to deduce the density deep below the crust. To a very good approximation, the earth is spherically symmetric, and so we may record the density of the earth as a function  $D(r)$  of the \*radius\$  $r$  from the center. With the density function  $D$  in hand, the mass of the earth  $E$  is given by the triple integral

$$M = \iiint_E D \, dV$$

*The true shape of the earth is called the Geoid, and is a slightly bumpy, slightly flattened sphere. But the total deviations from perfect spherical symmetry are less than 40km across the entire earth of diameter 12,760km: so a perfectly spherical model is more than sufficient for calculations such as mass and density!*

While the true density function of the earth is rather complicated, its general behavior is well-approximated by the following function: its more dense in the center at the metal core, with density dropping off to an approximate constant level in the mantle and surface (where the earth is mostly rock, not metal):

$$D(r) = a + \frac{b}{1 + r^2}$$

Where  $r$  is measured in *thousands of kilometers* and  $a$  is the density of rock ( $a = 3,000\text{kg}/\text{m}^3$ ), and  $b$  is the difference in density between rock and metal, or  $b = 6,000\text{kg}/\text{m}^3$ . For the coming calculations, its useful to compute these to the units of *kilograms per cubic 1000km* to match the units on  $r$ : That is  $a = 3 \cdot 10^{12}$  and  $b = 6 \cdot 10^{12}$ .

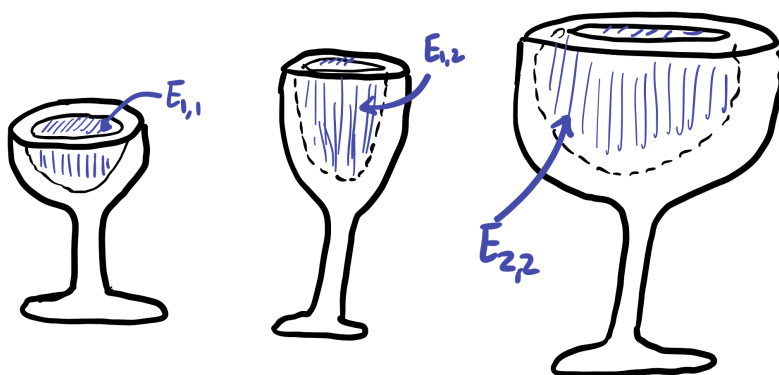


Figure 17.46.: Parametric design allows multiple different objects to be expressed using the same formula, just changing the constant parameters.

Approximating the earth as a sphere of radius  $R = 6,000$  kilometers, set up a triple integral in spherical coordinates that finds the mass of the earth.

Solve this integral (don't plug in the numbers until the end, as the formula will be much cleaner if you keep  $R, a, b$  as variables), and then plug in the relevant constants. What is the mass of the earth in kilograms? (Plug the numbers in using a calculator or computer: the result will be a very large number, best reported in scientific notation!)

## Sailing in the Bay

Two boats are sailing from one side of the bay to the other. In-between, the ocean water is flowing in a complicated pattern (due to tides flowing in and out of the golden gate).

Assignment 9

This water flow can be well-approximated by the vector field  $\vec{W}$

$$\vec{W} = \langle 2xy - y, x^2 - x + 1 \rangle$$

The first boat follows the path  $\vec{r}$  and the second follows the path  $\vec{s}$

$$\vec{r}(t) = \langle t, t^2 \rangle \quad \vec{s}(t) = \langle t^2, t^3 \rangle \quad 0 \leq t \leq 1$$

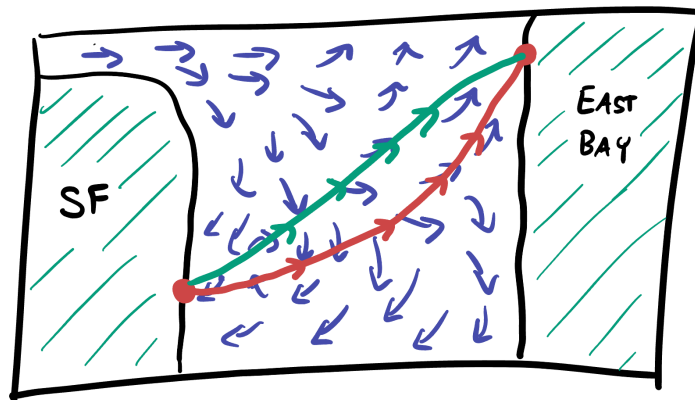


Figure 17.47.: Boats sailing across the SF bay.

Compute the appropriate line integrals to answer the following question: does one boat have a more difficult journey across the bay, or is the trip equally difficult for both vessels?